

LINEAR LAURENT PHENOMENON ALGEBRAS

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ABSTRACT. In [LP] we introduced Laurent phenomenon algebras, a generalization of cluster algebras. Here we give an explicit description of Laurent phenomenon algebras with a linear initial seed arising from a graph. In particular, any graph associahedron is shown to be the dual cluster complex for some Laurent phenomenon algebra.

1. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky in [CA1]. They were quickly recognized to be rather ubiquitous throughout mathematics, appearing for example in representation theory of quivers and finite-dimensional algebras, Poisson geometry, Teichmüller theory, integrable systems, and the study of Donaldson-Thomas invariants.

In [LP], we introduced *Laurent phenomenon algebras* (LP algebras), a generalization of cluster algebras where exchange polynomials were allowed to have arbitrarily many monomials, rather than being just binomials. The aim of this article is to describe certain LP algebras with a seed where the exchange polynomials are linear. These LP algebras give a large family of examples of finite type LP algebras of arbitrarily large rank.

One of the highlights of Fomin and Zelevinsky's theory of cluster algebras is the classification [CA2] of finite type cluster algebras, which turns out to be identical to the Cartan-Killing classification of semisimple Lie algebras. The combinatorial structure of finite type cluster algebras is captured by remarkable polytopal-complexes known as *generalized associahedra* [CFZ, FZ].

The cluster complexes of the LP algebras we study here are not known to be polytopal, but they contain subcomplexes which are polytopal and studied previously: the *graph associahedra* and *nestohedra* studied in [CD, FS, Pos, Zel]. In particular, any nestohedron arising from a directed graph is the exchange polytope of some LP algebra. This explains and confirms the “striking similarity” Zelevinsky has observed between nested complexes and cluster complexes [Zel].

We now describe the results of this paper in more detail.

1.1. Laurent phenomenon algebras. Let R be a coefficient ring containing algebraically independent elements A_1, A_2, \dots, A_n . For example, R could be $\mathbb{Z}[A_1, \dots, A_n]$. Let $\mathcal{F} = \text{Frac}(R[X_1, X_2, \dots, X_n])$ be the ambient field, where X_1, \dots, X_n are indeterminates.

A Laurent phenomenon algebra is a subring $\mathcal{A} \subset \mathcal{F}$ together with a collection of seeds $t = (\mathbf{x}, \mathbf{F})$, where $\mathbf{x} = \{x_1, \dots, x_n\} \subset \mathcal{F}$ is a set of variables, and $\mathbf{F} = \{F_1, \dots, F_n\} \subset R[x_1, \dots, x_n]$ is a set of exchange polynomials, satisfying conditions that shall be recalled in Section 2. The collection of seeds are connected by mutation. For each $i \in [n] =$

Date: March 8, 2013.

T.L. was supported by NSF grant DMS-0901111 and by a Sloan Fellowship.

P.P. was supported by NSF grant DMS-0757165.

$\{1, 2, \dots, n\}$, and a seed $t = (\mathbf{x}, \mathbf{F})$, we have a mutated seed $t' = \mu_i(t) = (\mathbf{x}', \mathbf{F}')$. The cluster variables of t and t' are the same except for x_i and x'_i , which satisfy an exchange relation $x_i x'_i = \hat{F}_i$, where \hat{F}_i is the exchange Laurent polynomial associated to F_i . For the reader comfortable with cluster algebras, the key difference is that \hat{F}_i is a Laurent polynomial rather than a non-Laurent binomial.

The *cluster complex* of a LP algebra is the simplicial complex with base set equal to the set of cluster variables, and faces corresponding to collections of cluster variables that lie in the same cluster. The *exchange graph* of a LP algebra \mathcal{A} is the graph with vertex set equal to the set of seeds of \mathcal{A} , and edges given by mutations.

1.2. Nested collections. Let Γ be a directed unweighted graph on $[n]$. A non-empty subset $I \subset [n]$ is *strongly connected* if the induced subgraph of Γ on I is strongly connected; that is, there is a directed path between every pair of vertices in I . We let $\mathcal{I} \subset 2^{[n]}$ denote the collection of strongly-connected subsets. A family of subsets $\mathcal{S} = \{I_1, \dots, I_k\} \in \mathcal{I}$ is *nested* if

- for any pair I_i, I_j either one of them lies inside the other, or they are disjoint;
- for any tuple of disjoint I_j -s, they are the strongly connected components of their union.

See Example 3.1 for an example. The *support* $S \subset [n]$ of a nested family \mathcal{S} is given by $S := \bigcup_k I_k$. Denote by $\mathcal{T} \subset \mathcal{I}$ the set of strongly connected components of Γ . If Γ is strongly connected, then $\mathcal{T} = \{[n]\}$.

The nested families with maximal support ($S = [n]$) form a simplicial complex $\mathcal{N} = \mathcal{N}(\Gamma)$, called the *nested set complex* by Feichtner and Sturmfels [FS] and the *nested complex* by Postnikov [Pos]. The base set of \mathcal{N} is the set of $\mathcal{I} \setminus \mathcal{T}$, and a subset of $\mathcal{I} \setminus \mathcal{T}$ is a face if it is nested. These complexes were studied in a more general context in [FS, Pos]. To each such simplicial complex $\mathcal{N}(\Gamma)$, there is an associated polytope $P(\Gamma)$, called a *nestohedron*, whose face lattice is dual to the nested complex. When Γ is an undirected graph, the nestohedron is known as a *graph associahedron*, studied by Carr and Devadoss [CD]. This family of polytopes includes the well-known associahedron and cyclohedron.

The *extended nested complex* $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}(\Gamma)$ is the simplicial complex on \mathcal{I} consisting of all nested families.

1.3. Acyclic functions. We define certain generating functions of acyclic subgraphs in Γ . For each non-empty subset $I \in 2^{[n]}$ let us consider functions $f: I \rightarrow [n]$ such that $i \rightarrow f(i)$ is an edge of Γ for each i , with the additional possibility that $f(i) = i$. Call such a function *acyclic* if the only directed cycles in the resulting graph are loops $i \mapsto f(i)$.

For a strongly-connected component $I \in 2^{[n]}$ define the following Laurent polynomials in X_1, X_2, \dots, X_n :

$$Y_I = \frac{\sum_{\text{acyclic } f: I \rightarrow [n]} \prod_{i \in I} \tilde{X}_{f(i)}}{\prod_{i \in I} X_i}$$

where

$$\tilde{X}_{f(i)} = \begin{cases} X_{f(i)} & \text{if } i \neq f(i) \\ A_{f(i)} & \text{if } i = f(i). \end{cases}$$

An example is given in Example 4.1, and further details on the Y_I rational functions are given in Section 4.

1.4. Graph LP algebras. Given a directed graph Γ , we define an initial seed t_Γ (also denoted t_\emptyset in Section 5) with cluster variables $\{X_1, \dots, X_n\}$ and exchange polynomials $F_i = A_i + \sum_{i \rightarrow j} X_j$, where $i \rightarrow j$ denotes an edge in Γ . Let $\mathcal{A}_\Gamma = \mathcal{A}(t_\Gamma)$ denote the LP algebra generated by the initial seed t_Γ . Our main theorem is

Theorem 1.1. *The normalized LP algebra $\mathcal{A}_\Gamma = \mathcal{A}(t_\Gamma)$*

- *has cluster variables*

$$\{X_1, X_2, \dots, X_n\} \cup \{Y_I \mid I \in \mathcal{I} \text{ is strongly connected}\};$$

- *has as clusters the sets of n cluster variables of the form*

$$\{X_{i_1}, X_{i_2}, \dots, X_{i_k}\} \cup \{Y_S \mid S \in \mathcal{S}\}$$

where \mathcal{S} is a maximal nested collection for $[n] \setminus \{i_1, i_2, \dots, i_k\}$;

- *has cluster complex the extended nested complex of Γ .*

Example 1.2. Consider a graph Γ on $[4] = \{1, 2, 3, 4\}$ with edges $1 \rightarrow 2, 2 \rightarrow 1, 1 \rightarrow 3, 3 \rightarrow 1, 3 \rightarrow 2, 2 \rightarrow 3, 1 \rightarrow 4, 3 \rightarrow 4, 4 \rightarrow 2$, as shown in Figure 1. Then the initial seed is given by

$$t_\Gamma = \{(X_1, A_1 + X_2 + X_3 + X_4), (X_2, A_2 + X_1 + X_3), (X_3, A_3 + X_1 + X_2 + X_4), (X_4, A_4 + X_2)\}.$$

The graph Γ has 11 strongly-connected subsets: $\emptyset, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}$ are the subsets which are *not* strongly-connected. Thus \mathcal{A}_Γ has 15 cluster variables

$$X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4, Y_{12}, Y_{23}, Y_{13}, Y_{123}, Y_{124}, Y_{234}, Y_{1234}$$

It also has 46 clusters, including:

$$\{X_1, X_2, X_3, X_4\}, \{X_1, Y_2, X_3, X_4\}, \{X_1, Y_2, X_3, Y_4\}, \{Y_{124}, Y_2, X_3, Y_4\}, \{Y_{124}, Y_2, Y_{1234}, Y_4\}, \dots$$

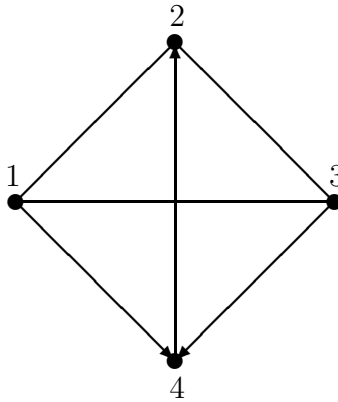


FIGURE 1. The example graph

Remark 1.3. It follows from the Laurent phenomenon property (Theorem 2.2) that each cluster variable X_i or Y_I is a Laurent polynomial when expressed in terms of the variables of any cluster.

1.5. Exchange relations. The exchange polynomials $F_i^{\mathcal{S}}$ of the seed $t_{\mathcal{S}}$ associated to a nested collection \mathcal{S} are described in Section 5. Roughly speaking, the exchange polynomials count certain paths in Γ , when expressed not in terms of the cluster variables of $t_{\mathcal{S}}$, but in terms of all the Y_I . The exchange relation turns out to have an interpretation as a determinantal identity known as *Dodgson's condensation*.

Example 1.4. In the example in Figure 1 take the cluster $\{Y_1, Y_{12}, Y_{123}, X_4\}$. The exchange relations in this cluster are

$$Y_1 Y_2 = 1 + Y_{12}, \quad Y_{12} Y_{13} = 1 + Y_1^2 + Y_1(2 + Y_{123}),$$

$$\begin{aligned} Y_{123} X_3 &= \frac{X_4(1 + Y_1)(1 + Y_{12}) + A_1(1 + Y_1 + Y_{12}) + A_2 Y_1(1 + Y_1) + A_3 Y_1 Y_{12}}{Y_1} \\ &= X_4(Y_{12} + Y_2 + 1) + A_1(1 + Y_2) + A_2(1 + Y_1) + A_3 Y_{12}, \end{aligned}$$

and

$$\begin{aligned} X_4 Y_{1234} &= \frac{1}{Y_1 Y_{12}} (A_1(1 + Y_{12} + Y_1^2 + Y_1(2 + Y_{123}) + Y_1 Y_{12}) \\ &\quad + A_2(Y_1^3 + Y_1^2(2 + Y_{123}) + Y_1) + A_3(Y_1^2 Y_{12} + Y_1 Y_{12}) + A_4 Y_1 Y_{12} Y_{123}) \\ &= A_1(Y_3 + 1) + A_2 Y_{13} + A_3(Y_1 + 1) + A_4 Y_{123} \end{aligned}$$

1.6. Graph associahedra as cluster complexes of LP algebras. Let \mathcal{A}'_{Γ} be the LP algebra obtained from \mathcal{A}_{Γ} by freezing (see [LP]) all the variables $\{Y_T \mid T \in \mathcal{T}\}$. In more detail, consider the subset of seeds of \mathcal{A}_{Γ} which contain all the cluster variables $\{Y_T \mid T \in \mathcal{T}\}$. Removing all the Y_T -s from these seeds, we obtain the LP algebra \mathcal{A}'_{Γ} , where now the coefficient ring is $R[Y_T^{\pm 1} \mid T \in \mathcal{T}]$, but the ambient field \mathcal{F} is the same. Note that none of the X_i is a cluster variable in \mathcal{A}'_{Γ} , and the rank of \mathcal{A}'_{Γ} is equal to $n - |\mathcal{T}|$.

As a consequence of Theorem 1.1, we have

Corollary 1.5. *The cluster complex of \mathcal{A}'_{Γ} is the nested set complex of Γ . The exchange graph of \mathcal{A}'_{Γ} is the one-skeleton of the (di)graph associahedron $P(\Gamma)$ associated to Γ .*

In the case that Γ is an undirected path, $P(\Gamma)$ is the usual *associahedron*. As explained in Section 6, in this case \mathcal{A}'_{Γ} is in fact identical to the cluster algebra of type A_n , which is known to have cluster complex dual to the associahedron.

In the case that Γ is a cycle, $P(\Gamma)$ is the *cyclohedron*. However, \mathcal{A}'_{Γ} is not the cluster algebra of type B_n , though it does have the same exchange graph.

We conjecture (Conjecture 7.6) that cluster complexes of all linear LP algebras are polytopal.

1.7. General linear LP algebras. The exchange polynomials $F_i = A_i + \sum_{i \rightarrow j} X_j$ of our initial seed are not the most general linear polynomials that we could choose for a seed of a LP algebra. Let us call a LP algebra \mathcal{A} a *linear LP algebra* if it has a seed such that all exchange polynomials are linear in the cluster variables of that seed. We conjecture (Conjecture 7.5) that all linear LP algebras are of finite type.

If the A_i are not algebraically independent, but satisfy relations, or for example are set to 0, then the corresponding linear LP algebra appears to be a degeneration of the graph LP algebra, and in particular has fewer clusters.

Suppose the coefficients of X_j in F_i of a linear seed of a linear LP algebra \mathcal{A} are arbitrary elements of a coefficient ring R , and not just equal to 0 or 1, but the constant term is still an indeterminate A_i . One can still associate a graph Γ which has an edge $i \rightarrow j$ if X_j occurs in F_i with non-zero coefficient. The cluster complex of \mathcal{A} appears to be the same as that of \mathcal{A}_Γ studied here, though the coefficients of the exchange polynomials exhibit interesting dynamics. This phenomenon is illustrated in the classification of rank 2 LP algebras in [LP]. The most interesting rank 2 linear case is that associated to the complete undirected graph on 2 vertices, with a pentagonal exchange graph.

1.8. Positivity and other conjectures. The LP algebras \mathcal{A}_Γ appear to satisfy many positivity and integrality properties beyond the formulae proven in this paper. We list some conjectures in Section 7.

2. BACKGROUND ON LAURENT PHENOMENON ALGEBRAS

We recall background on LP algebras from [LP].

2.1. Seeds. Let R be a *coefficient ring* over \mathbb{Z} , which we assume to be a unique factorization domain. For example R could be \mathbb{Z} , a polynomial ring over \mathbb{Z} , or a Laurent polynomial ring over \mathbb{Z} . Let $n \geq 1$ be a positive integer and write $[n]$ for $\{1, 2, \dots, n\}$. Let the *ambient field* \mathcal{F} be the rational function field in n independent variables over the field of fractions $\text{Frac}(R)$.

A *seed* in \mathcal{F} is a pair (\mathbf{x}, \mathbf{F}) where

- $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is a transcendence basis for \mathcal{F} over $\text{Frac}(R)$.
- $\mathbf{F} = \{F_1, F_2, \dots, F_n\}$ is a collection of polynomials in $\mathcal{P} = R[x_1, x_2, \dots, x_n]$ satisfying:
 - (LP1) F_i is an irreducible element of \mathcal{P} and is not divisible by any variable x_j
 - (LP2) F_i does not involve the variable x_i

The variables $\{x_1, x_2, \dots, x_n\}$ are called *cluster variables*, and the polynomials

$$\{F_1, F_2, \dots, F_n\}$$

are called *exchange polynomials*. As is usual in the theory of cluster algebras, the set $\{x_1, x_2, \dots, x_n\}$ will be called a *cluster*. If $t = (\mathbf{x}, \mathbf{F})$ is a seed, we let $\mathcal{L} = \mathcal{L}(t)$ denote the Laurent polynomial ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. If x is a cluster variable, we shall use the notation F_x to denote the exchange polynomial associated to a cluster variable x . We call n the *rank* of the seed (\mathbf{x}, \mathbf{F}) . The set $\{x_1, \dots, x_n\}$ of cluster variables in a seed will be called a *cluster*.

For each seed (\mathbf{x}, \mathbf{F}) , we define a collection $\{\hat{F}_1, \hat{F}_2, \dots, \hat{F}_n\} \subset \mathcal{L}$ of Laurent polynomials by the conditions:

- $\hat{F}_j = x_1^{a_1} \cdots \hat{x}_j \cdots x_n^{a_n} F_j$ for some $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n \in \mathbb{Z}_{\leq 0}$
-

$$(2.1) \quad \hat{F}_i|_{x_j \leftarrow F_j/x} \in R[x_1^{\pm 1}, \dots, x_{j-1}^{\pm 1}, x_j^{\pm 1}, x_{j+1}^{\pm 1}, \dots, x_n^{\pm 1}] \text{ and is not divisible by } F_j$$

The divisibility is to be checked in $R[x_1^{\pm 1}, \dots, x_{j-1}^{\pm 1}, x_j^{\pm 1}, x_{j+1}^{\pm 1}, \dots, x_n^{\pm 1}]$.

The \hat{F} are well-defined ([LP, Lemma 2.3]). Furthermore, the collections $\{F_1, \dots, F_n\}$ and $\{\hat{F}_1, \dots, \hat{F}_n\}$ determine each other uniquely ([LP, Lemma 2.3]).

Suppose $i \in [n]$. Then we say that a tuple $(\mathbf{x}', \mathbf{F}')$ is obtained from a seed (\mathbf{x}, \mathbf{F}) by mutation at i , and write $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$, if the former can be obtained from the latter by the following procedure.

The cluster variables of $\mu_i(\mathbf{x}, \mathbf{F})$ are given by $x'_i = \hat{F}_i/x_i$ and $x'_j = x_j$ for $j \neq i$. The exchange polynomials $F'_j \in \mathcal{L}'$ are obtained from F_j as follows. If F_j does not depend on x_i , then we must have F'_j/F_j be a unit in R , where now F'_j is considered as an element of \mathcal{L}' . Otherwise $\hat{F}_i(0)$ is well defined. We define G_j by

$$(2.2) \quad G_j = F_j \Big|_{x_i \leftarrow \frac{\hat{F}_i|_{x_j \leftarrow 0}}{x'_i}}$$

Next, we define H_j to be G_j with all common factors (in $R[x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n]$) with $\hat{F}_i|_{x_j \leftarrow 0}$ removed. Note that this defines H_j only up to a unit in R . Finally we have $F'_j = MH_j$ where M is a Laurent monomial in the $x'_1, x'_2, \dots, \hat{x}'_j, \dots, x'_n$ with coefficient in R , such that $F'_j \in \mathcal{P}'$, satisfies (LP2), and is not divisible by any variable in \mathcal{P}' . For any H_j , it is always possible to pick the monomial M to satisfy these conditions, but in general there are many choices for the coefficient of M . In particular F'_j is defined only up to a unit in R .

In [LP, Proposition 2.9] it was shown that if $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$ is obtained by mutation of (\mathbf{x}, \mathbf{F}) at i then $(\mathbf{x}', \mathbf{F}')$ is also a seed. It is clear that if \mathbf{x} is a transcendence basis of \mathcal{F} over $\text{Frac}(R)$, then so is \mathbf{x}' . It was shown in [LP, Proposition 2.10] that mutations are involutions.

Remark 2.1. By definition, the mutated exchange polynomial F'_i is obtained from F_i by the substitution $x_k \leftarrow \hat{F}_k|_{x_i=0}/x'_k$. We will often use the fact that sometimes we can perform this substitution in \hat{F}_i instead of F_i , with the answer only being off a monomial factor. This is the case when \hat{F}_i/F_i does not involve x_k .

2.2. Laurent phenomenon algebras. Let R be a fixed coefficient ring and \mathcal{F} denote the ambient fraction field in n indeterminates as in Section 2.1. A *Laurent phenomenon algebra* $(\mathcal{A}, \{(\mathbf{x}, \mathbf{F})\})$ is a subring of $\mathcal{A} \subset \mathcal{F}$ together with a distinguished collection of seeds $\{(\mathbf{x}, \mathbf{F})\} \subset \mathcal{F}$ belonging to the ambient field \mathcal{F} . The algebra $\mathcal{A} \subset \mathcal{F}$ is generated over R by all the variables \mathbf{x} in any of the seeds of \mathcal{A} . The seeds satisfy the condition: for each seed (\mathbf{x}, \mathbf{F}) and $i \in [n]$, we are given a seed $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$ obtained from (\mathbf{x}, \mathbf{F}) by mutation at i . Thus the seeds form the vertices of a n -regular graph, where the edges are mutations. Furthermore, we assume all seeds are connected by mutation.

If $t = (\mathbf{x}, \mathbf{F})$ is any seed in \mathcal{F} , we shall let $\mathcal{A}(t)$ denote any LP algebra which has t as a seed. We say that $\mathcal{A}(t)$ is generated (as a LP algebra) by t , or has *initial seed* t . Since seed mutation is only well-defined up to units, the seeds of $\mathcal{A}(t)$ are not determined by t .

We say that two seeds (\mathbf{x}, \mathbf{F}) and $(\mathbf{x}', \mathbf{F}')$ are *equivalent* if the following two conditions hold:

- (1) For each i we have x_i/x'_i is a unit in R , and

- (2) For each i we have F_i/F'_i is a unit in R , where F_i and F'_i are both considered as elements of the ambient field

$$\mathcal{F} = \text{Frac}(R[x_1, x_2, \dots, x_n]) = \text{Frac}(R[x'_1, x'_2, \dots, x'_n]).$$

Let \mathcal{A} be a Laurent phenomenon algebra. We will say that \mathcal{A} is *normalized* if whenever two seeds t_1, t_2 are equivalent, we have that $t_1 = t_2$. Suppose \mathcal{A} is a LP algebra which is normalized. Then we say that \mathcal{A} is of *finite type* if it has finitely many seeds. All the LP algebras we study in this paper will be normalized and of finite type.

2.3. Some properties of LP algebras.

Theorem 2.2. [LP, Theorem 5.1] *The Laurent phenomenon holds: every cluster variable of a LP algebra \mathcal{A} is a Laurent polynomial in the cluster variables of any seed t of \mathcal{A} .*

Lemma 2.3. *Let $t = (\mathbf{x}, \mathbf{F})$ be a seed of a LP algebra $\mathcal{A}(t)$, and x_i one of the cluster variables of t . Suppose that $m = \prod_{j \neq i} x_j^{d_j}$ is a monomial in other cluster variables of t such that F_i/m can be expressed as a polynomial in the cluster variables (not necessarily belonging to t) of \mathcal{A} . Suppose $F_m/\hat{F}_m = \prod_{j \neq i} x_j^{d'_j}$. Then we have $d'_j \geq d_j$.*

Proof. By Theorem 2.2, being a polynomial in some cluster variables, F_i/m must be a Laurent polynomial in any cluster of this LP algebra. This implies that $F_i/m \in \mathcal{L}(t) \cap \mathcal{L}(\mu_j(t))$. Comparing with (2.1), we obtain the desired inequality. \square

Theorem 2.4. *Let $t = (\mathbf{x}, \mathbf{F})$ be a seed and $i \neq j$ be two indices be such that x_j does not appear in F_i and such that F_i/F_j is not a unit in R . Then the mutations at i and j commute. More precisely, we can choose seed mutations so that*

$$\mu_i(\mu_j(t)) = \mu_j(\mu_i(t)).$$

The proof of Theorem 2.4 is postponed until Section 8.

2.4. Notation. For $f, g \in \mathcal{L}(t)$ we shall write $f \propto g$ to mean that f and g are equal up to a unit in $\mathcal{L}(t)$. That is, f and g differ multiplicatively by a (Laurent) monomial factor $r \prod_i x_i^{a_i}$ in the cluster variables of t , where $r \in R$ is a unit in R . Note that this differs from the usage of \propto in [LP].

3. NESTED COMPLEXES

For a set S and elements i, j, k (usually not in S), we will use the concatenations $Si, Sij, Sijk$ to denote the unions $S \cup \{i\}$, $S \cup \{i, j\}$ and $S \cup \{i, j, k\}$. Occasionally we will also concatenate subsets and elements such as $SiTj$ to indicate the union $S \cup \{i\} \cup T \cup \{j\}$.

3.1. Maximal nested collections. Let Γ be a directed loopless multiplicity-free graph on the vertex set $[n] = \{1, 2, \dots, n\}$. Thus, every edge $i \rightarrow j$ is either present with multiplicity one or absent, for each ordered pair (i, j) , $i \neq j$. We identify subsets of vertices of Γ with the corresponding induced subgraphs, for example we shall talk about *strongly connected* subsets, and so on. Recall that we defined nested collections $\mathcal{S} = \{S_1, S_2, \dots, S_k\} \subset \mathcal{I}$ in Section 1.2. We say that a nested collection \mathcal{S} is *maximal* for $S = \bigcup_i S_i$ if for any other nested collection \mathcal{S}' with $\bigcup_i S'_i = S$ we have $\mathcal{S} \subset \mathcal{S}'$ implies $\mathcal{S} = \mathcal{S}'$. A nested collection \mathcal{S} is maximal if it is maximal for its support.

Example 3.1. Consider the graph in Example 1.2. The following are some of the families of nested subsets:

$$\begin{aligned} & \{\{3\}, \{4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}, \\ & \{\{3\}, \{4\}, \{1, 3\}, \{1, 2, 3, 4\}\}, \\ & \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}. \end{aligned}$$

All of these families are maximal.

The following result can for example be found in [Pos, Proposition 7.6].

Lemma 3.2. *Suppose $\mathcal{S} = \{S_i\}_{i=1}^k$ is a maximal nested collection for $S = \bigcup_i S_i$. Then there is a bijection $\phi : S \rightarrow \mathcal{S}$ given by the condition that $\phi(s)$ is the smallest subset in \mathcal{S} containing s .*

We will always assume that the subsets are indexed so that the bijection ϕ is the identity: that is, S_i is the smallest subset containing i . Lemma 3.2 gives rise to a partial order $\preceq_{\mathcal{S}}$ on S : we have $i \preceq_{\mathcal{S}} j$ if and only if $S_i \subseteq S_j$. Furthermore, this partial order uniquely determines \mathcal{S} . For any non-maximal $i \in S$, there is a unique $j \in S$ which covers i in this partial order. We denote this element by i^+ . We denote the maximal elements of S by S^{\max} .

Let \mathcal{S} be a maximal nested collection. Suppose $k \in S^{\max}$ is maximal. Then removing S_k from \mathcal{S} gives a legitimate nested collection \mathcal{S}^k , which is a maximal nested collection for $S - \{k\}$. Also let $k \in S$ be arbitrary. Then define \mathcal{S}_k by removing from \mathcal{S} all subsets $S \supsetneq S_k$. If $k \in S^{\max}$, then $\mathcal{S}_k = \mathcal{S}$.

For $S \subset [n]$ denote by $S \oplus j$ the strongly connected component of Sj that contains j . Denote by $S \ominus j = Sj - (S \oplus j)$ the elements of S that are not included in $S \oplus j$. Note that these definitions are made both for $j \in S$ and for $j \notin S$.

Example 3.3. Consider the maximal nested collection $\mathcal{S} = \{3\}, \{4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$ from Example 3.1. Then

$$\phi(1) = \{1, 2, 3, 4\}, \quad \phi(2) = \{2, 3, 4\}, \quad \phi(3) = \{3\}, \quad \phi(4) = \{4\}.$$

So for the indexing associated with ϕ we have $S_2 = \{2, 3, 4\}$, and so on. The corresponding partial order is given by transitive closure of the relation $3 \preceq_{\mathcal{S}} 2$, the relation $4 \preceq_{\mathcal{S}} 2$ and the relation $2 \preceq_{\mathcal{S}} 1$. We also have $3^+ = 4^+ = 2$, $2^+ = 1$ and $\mathcal{S}_2 = \{3\}, \{4\}, \{2, 3, 4\}$, $\mathcal{S}_3 = \{3\}, \{4\}$. Finally, we have $\{3, 4\} \oplus 1 = \{1, 3\}$, $\{3, 4\} \ominus 1 = \{4\}$, $\{3, 4\} \oplus 2 = \{2, 3, 4\}$, $\{3, 4\} \ominus 2 = \emptyset$.

3.2. Mutation of nested collections. We describe a way of building maximal nested collections from the empty nested collection. Suppose \mathcal{S} is a maximal nested collection with support S and $s \notin S$. We define $\mu_s(\mathcal{S}) = \mathcal{S} \cup \{S \oplus s\}$, which has support $S \cup \{s\}$. This is a kind of *mutation* of \mathcal{S} which we call *activation* at s . Given an ordered subset $\vec{S} = (s_1, s_2, \dots, s_k) \subset [n]$, we define $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_k = \mathcal{S}$ as follows. We have $\mathcal{S}_0 = \emptyset$ and recursively define $\mathcal{S}_{r+1} = \mu_{s_{r+1}}(\mathcal{S}_r)$.

Lemma 3.4. *The activation $\mu_s(\mathcal{S})$ for $s \notin S$ is a maximal nested collection on $S \cup \{s\}$. Any maximal nested collection can be obtained by activating a sequence $\vec{S} = (s_1, s_2, \dots, s_k)$.*

Proof. The fact that $\mu_s(\mathcal{S})$ is nested is checked directly from the definition, and maximality follows from counting. For the second statement, recall that maximal nested collections \mathcal{S} have a partial order $\preceq_{\mathcal{S}}$, given by the ordering $i \preceq_{\mathcal{S}} j$ if $S_i \subset S_j$. Now, any sequence (s_1, s_2, \dots, s_k) of activations that agree with the partial order, that is, satisfies $s_r \preceq_{\mathcal{S}} s_{r'} \implies r \leq r'$, yield the desired \mathcal{S} . \square

Example 3.5. Consider the activation sequence $(3, 4, 2, 1)$ for the graph in Example 1.2. Using

$$\emptyset \oplus 3 = \{3\}, \quad \{3\} \oplus 4 = \{4\}, \quad \{3, 4\} \oplus 2 = \{2, 3, 4\}, \quad \{2, 3, 4\} \oplus 1 = \{1, 2, 3, 4\}$$

we see that the result is the maximal nested collection $\{\{3\}, \{4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. If on the other hand we used the activation sequence $(3, 4, 1, 2)$, we would end up with $\{3\}, \{4\}, \{1, 3\}, \{1, 2, 3, 4\}$.

Let $\vec{S} = (s_1, \dots, s_k)$ be a sequence of activations. We call a pair of activations s_i and s_{i+1} *exchangeable* if S_{s_i} and $S_{s_{i+1}}$ are disjoint.

Lemma 3.6. *If we swap an exchangeable pair of activations in \vec{S} , the resulting collection \mathcal{S} does not change. Furthermore, any two sequences of activations that yield the same \mathcal{S} can be transformed one into another by swapping exchangeable pairs.*

Proof. The first claim is clear, since if s_i and s_{i+1} are not strongly connected via (s_1, \dots, s_{i-1}) , it does not matter which one of the two we activate first. For the second claim, observe that Lemma 3.4 gives a bijection between activation sequences yielding \mathcal{S} and linear extensions of $\preceq_{\mathcal{S}}$. Thus our statement is equivalent to the following combinatorial statement: any two linear extensions of a poset P can be connected by successively swapping the positions of i and $i + 1$, while always staying a linear extension. This is a simple combinatorial fact that can easily be verified. \square

So far we have discussed activating an element (or vertex) $s \notin S$. Now we describe the mutation $\mu_s(\mathcal{S})$ for $s \in S$. If $s \in S^{\max}$ then $\mu_s(\mathcal{S}) = \mathcal{S}^s$ is obtained from \mathcal{S} by removing S_s . In this case we can say that s has been *deactivated* and $\mu_s(\mathcal{S})$ is a maximal nested collection with support $\mathcal{S} - \{s\}$. If $s \notin S^{\max}$, we define $\mu_s(\mathcal{S})$ to be the collection obtained from \mathcal{S} by removing S_s and adding the new subset $S_{s+} - s \oplus s^+$. We call this operation *internal mutation* at s .

Example 3.7. Take the maximal collection $\{3\}, \{4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$, and take $s = 2$. Then $2^+ = 1$, $S_1 - 2 = \{1, 2, 3, 4\} - 2 = \{1, 3, 4\}$ and $\{1, 3, 4\} \oplus 1 = \{1, 3\}$. Thus, we remove $S_2 = \{2, 3, 4\}$ from the family, and replace it by $\{1, 3\}$, thus obtaining the new collection $\{3\}, \{4\}, \{1, 3\}, \{1, 2, 3, 4\}$.

Lemma 3.8. *The internal mutation $\mu_s(\mathcal{S})$ for $s \in S - S^{\max}$ is a maximal nested collection on S . Furthermore, $\mu_s(\mathcal{S})$ is the unique maximal nested collection with the same support as \mathcal{S} , other than \mathcal{S} , which contains all the subsets $\{S_r \mid r \neq s\}$.*

Proof. Maximality and the claim about the support of $\mathcal{S}' = \mu_s(\mathcal{S})$ are clear. We check that \mathcal{S}' is nested. First note that we have $S'_s = S_{s+}$ and $S'_{s+} = S_{s+} - s \oplus s^+$.

Suppose $S_r \in \mathcal{S}'$ and $S'_{s+} = S_{s+} - s \oplus s^+$ intersect non-trivially. Then S_r intersects S_{s+} non-trivially, and so must be contained in S_{s+} . But then it is clear from the definition of S'_{s+} that either $S_r \subset S'_{s+}$ or $s \in S_r$. In the latter case we must have $S_r = S_{s+}$.

Now suppose we have a disjoint collection $S_{r_1}, S_{r_2}, \dots, S_{r_k}, S'_{s^+} \in \mathcal{S}'$. We need to show that they are the strongly connected components of their union. The only possible way this can fail is if we can find i_1, i_2, \dots, i_j so that $S_{r_{i_1}}, S_{r_{i_2}}, \dots, S_{r_{i_j}}$ and S'_{s^+} have a strongly connected union which is a subset of S_{s^+} . This is impossible since none of $S_{r_{i_1}}, S_{r_{i_2}}, \dots, S_{r_{i_j}}$ contain s , and the assumption that \mathcal{S} was nested.

For the last statement, suppose \mathcal{S}' is a maximal nested collection containing all the subsets $\{S_r \mid r \neq s\}$. For any $r \neq s, s^+$ it is clear that $\mathcal{S}'_r = S_r$, so by Lemma 3.2 we have that \mathcal{S}' is determined by whether we have $\mathcal{S}'_s = S_s$ or $\mathcal{S}'_s = S_{s^+}$. In the first case we have $\mathcal{S}' = \mathcal{S}$, and in the latter case we have $\mathcal{S}' = \mu_s(\mathcal{S})$. \square

3.3. The exchange graph on maximal nested collections. Let \mathfrak{C} denote the set of all maximal nested collections \mathcal{S} with support equal to any subset $S \subset [n]$. We define a n -regular graph with vertex set \mathfrak{C} by connecting \mathcal{S} and $\mu_s(\mathcal{S})$ by an edge labeled s . Note that we never have $\mu_s(\mathcal{S}) = \mathcal{S}$.

4. ACYCLIC FUNCTIONS

4.1. Acyclic functions and strongly connected components. We first give an example of the Y Laurent polynomial defined in Section 1.3.

Example 4.1. Consider the graph Γ of Example 1.2. Then we have

$$Y_{124} = \frac{X_1(X_2(X_3 + A_1) + A_4(X_3 + X_4 + A_1)) + (X_2 + A_4)(X_2 + X_3 + X_4 + A_1)(X_3 + A_2)}{X_1 X_2 X_4}.$$

For example, the term $X_2 X_3 A_2$ corresponds to the function $f(1) = 3, f(2) = 2, f(4) = 2$. The functions f that do *not* contribute to the numerator are the functions that have either $f(1) = 2, f(2) = 1$ (and $f(4) = 4$ or $f(4) = 2$), or the function $f(1) = 4, f(4) = 2, f(2) = 1$.

We continue to fix a multiplicity-free loopless directed graph Γ on $[n]$.

Lemma 4.2. *Let $I_1, \dots, I_k \in \mathcal{I}$ be the strongly connected components of I . Then $Y_I = \prod_{j=1}^k Y_{I_j}$.*

Proof. Any combination of acyclic functions on the I_j yields an acyclic function on I , since there is no cycle that can be created in between different strongly connected components of I . \square

Lemma 4.3. *We have $Y_{Si} = Y_{S \oplus i} Y_{S \ominus i}$.*

Proof. Clear from Lemma 4.2. \square

Lemma 4.4. *Suppose $I \in \mathcal{I}$ is strongly connected. Then the numerator of Y_I is irreducible in the ring of polynomials in X_i . Furthermore, these numerators are pairwise coprime.*

Proof. The proof is by induction on the size of I , and the base case $|I| = 1$ is clear. Assume the numerator of Y_I factors as PQ . Since it is multilinear in the $A_i, i \in I$, so must be each of the factors P and Q . Furthermore, each A_i appears either in P or in Q but not both. Let I_P and I_Q be the sets of i -s such that A_i appears in the corresponding factor. Let $j \in I_P$. Then since numerator of Y_I is linear in A_j , the coefficient of A_j must be divisible by Q . However, this coefficient is the numerator of $Y_{I - \{j\}}$. By induction assumption, and

from the fact that the polynomial ring is a unique factorization domain, it follows that Q must be product of numerators of Y_J for some strongly connected components J of $I - \{j\}$. Since this is true for any $j \in I_P$, and since each A_i , $i \notin I_P$ must appear in Q , we conclude that Q is the numerator of Y_{I_Q} . Similarly P is the numerator of Y_{I_P} . However, the identity $Y_I = Y_{I_P} Y_{I_Q}$ fails, since I is strongly connected and thus there are cycles that fail to be cycles when restricted to I_P or I_Q .

The last statement is easy: the numerator of Y_I has a monomial $\prod_{i \in I} A_i$ which occurs in no other numerator. \square

4.2. Acyclic functions as determinants. We shall use the notation $Y_i := Y_{\{i\}}$. Let $\mathfrak{Y} = (\mathfrak{y}_{ij})$ be the $n \times n$ matrix defined by

$$\mathfrak{y}_{ij} = \begin{cases} Y_i & \text{if } i = j; \\ -1 & \text{if there is an edge } i \longrightarrow j \text{ in } \Gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Denote \mathfrak{Y}_I the principal minor of \mathfrak{Y} with row and column indices I .

Proposition 4.5. *We have*

$$Y_I = \det(\mathfrak{Y}_I).$$

Proof. By the inclusion-exclusion principle, to count the acyclic functions from the set I (with weight assigned as above) we need to take the alternating sum over all collections of disjoint cycles in I of functions containing those cycles. In other words,

$$Y_I = \sum_{\text{collections } \mathfrak{C} \text{ of disjoint cycles}} (-1)^{\text{number of cycles in } \mathfrak{C}} Y_{I-\mathfrak{C}}.$$

This is exactly the expansion of the determinant of the minor \mathfrak{Y}_I . Indeed, the off-diagonal entries give a collection of disjoint cycles in Γ . Furthermore, the sign of a term in the determinant is $(-1)^{\text{number of even cycles}}$, while the sign obtained from having -1 entries off diagonal in \mathfrak{Y} is $(-1)^{\text{number of odd cycles}}$. Together, those give us the desired sign $(-1)^{\text{number of cycles}}$. \square

Example 4.6. For the graph of Example 1.2, we have

$$\mathfrak{Y} = \begin{pmatrix} \frac{A_1+X_2+X_3+X_4}{X_1} & -1 & -1 & -1 \\ -1 & \frac{A_2+X_1+X_3}{X_2} & -1 & 0 \\ -1 & -1 & \frac{A_3+X_1+X_2+X_4}{X_3} & -1 \\ 0 & -1 & 0 & \frac{A_4+X_2}{X_4} \end{pmatrix}.$$

It is easy to check that the variable Y_{124} in the example above is equal to the principal minor of \mathfrak{Y} with row and column set $\{1, 2, 4\}$.

4.3. Exchange relations. We describe what will turn out to be the exchange relations in the LP algebra associated with Γ . Let us use the notation $p : i \longrightarrow_S j$ to denote simple (vertex non-repeating) paths p from vertex i to vertex j such that all intermediate vertices along the path belong to the set S . The i and j vertices may or may not belong to S . Define

$$P_S^{i,j} = \sum_{p:i \longrightarrow_S j} Y_{S-p}.$$

Note that for $j = i$ the only simple path from i to i is trivial path, and thus $P_S^{i,i} = Y_S$ for $i \notin S$ and $P_S^{i,i} = Y_{S-\{i\}}$ for $i \in S$.

Lemma 4.7. *For $i \notin S$ we have*

$$X_i Y_{S \oplus i} = \frac{\sum_{j \notin Si} P_S^{i,j} X_j + \sum_{j \in Si} P_S^{i,j} A_j}{Y_{S \ominus i}}.$$

Proof. By Lemma 4.3 the desired equality is equivalent to

$$X_i Y_{Si} = \sum_{j \notin Si} P_S^{i,j} X_j + \sum_{j \in Si} P_S^{i,j} A_j.$$

For any acyclic function counted in the numerator of Y_{Si} , just follow the arrows from i until this path either leaves S or end with a loop in S . Group acyclic functions by the endpoints j of the resulting paths p . \square

Lemma 4.8. *The expression*

$$\frac{\sum_{j \notin Si} P_S^{i,j} X_j + \sum_{j \in Si} P_S^{i,j} A_j}{Y_{S \ominus i}}$$

can be written as a polynomial in the X_j , A_j and Y_T 's for $T \subset S$.

Proof. Since it is equal to $X_i Y_{S \oplus i}$, it depends only on $S \oplus i$ and not S itself. Thus we can compute it taking Lemma 4.7 with $S := S \oplus i$, in which case $S \ominus i = \emptyset$ so that there is no denominator. \square

Example 4.9. Continuing Example 4.1, Lemma 4.7 gives for $S = \{1, 2\}$, $i = 4$:

$$X_4 Y_{124} = \frac{X_3(Y_1 + 1) + A_4 Y_{12} + A_1 + A_2 Y_1}{1},$$

and for $S = \{3, 4\}$ and $i = 1$:

$$X_1 Y_{13} = \frac{X_2(Y_3 Y_4 + Y_3 + Y_4 + 1) + A_1 Y_3 Y_4 + A_4(Y_3 + 1) + A_3 Y_4}{Y_4}.$$

This last expression can be rewritten

$$\frac{X_2(Y_3 Y_4 + Y_3 + Y_4 + 1) + A_1 Y_3 Y_4 + A_4(Y_3 + 1) + A_3 Y_4}{Y_4} = \frac{X_2(Y_3 + 1) + X_4(Y_3 + 1) + A_1 Y_3 + A_3}{1}.$$

Lemma 4.10. *For $i \neq j$, and a set S we have*

$$P_S^{i,j} = (-1)^{1+\text{number of elements of } S \text{ between } i \text{ and } j} \det(\mathfrak{Y}_{S' i, S' j}),$$

where $S' = S - \{i, j\}$ and $\mathfrak{Y}_{I,J}$ denotes the minor of \mathfrak{Y} with row indices I and column indices J .

Proof. Swap columns of $\mathfrak{Y}_{Si, Sj}$ so that column j moves to the i -th position. This creates a sign $(-1)^{\text{number of elements of } S \text{ between } i \text{ and } j}$. In the expansion of the resulting determinant there is one cycle which corresponds to a path from i to j in S . Group terms according to this path. According to Proposition 4.5 we get exactly the summation in the definition of $P_S^{i,j}$, except we need to check that the signs work out correctly. The terms corresponding to a specific path p from i to j receive minus sign from the definition of the determinant in case the path has even length. They also receive minus sign from -1 entries of \mathfrak{Y} in

case p has odd length. Thus, all terms have the same minus sign, which contributes 1 in the exponent of $(-1)^{1+\text{number of elements of } S \text{ between } i \text{ and } j}$. \square

Recall that the *Jacobi identity*, also known as *Dodgson's condensation* states that for a square $n \times n$ matrix M we have

$$(4.1) \quad \det(M) \det(M_{[2,n-1],[2,n-1]}) = \det(M_{[1,n-1],[1,n-1]}) \det(M_{[2,n],[2,n]}) - \det(M_{[1,n-1],[2,n]}) \det(M_{[2,n],[1,n-1]}).$$

Lemma 4.11. *For $i, j \notin S$ with $i \neq j$ we have*

$$Y_{S \oplus i} Y_{S \oplus j} = \frac{Y_{Sij} Y_S + P_S^{i,j} P_S^{j,i}}{Y_{S \oplus i} Y_{S \oplus j}}.$$

Proof. By Lemma 4.3, the equality is equivalent to

$$Y_{Si} Y_{Sj} = Y_{Sij} Y_S + P_S^{i,j} P_S^{j,i}.$$

According to Lemma 4.10, this is just the identity (4.1). \square

Example 4.12. With the graph in Example 1.2, we have

$$P_{\{3,4\}}^{1,2} = Y_{34} + Y_3 + Y_4 + 1 = -\det \begin{pmatrix} -1 & -1 & -1 \\ -1 & Y_3 & -1 \\ -1 & 0 & Y_4 \end{pmatrix}$$

and $P_{\{3,4\}}^{2,1} = Y_{34} + Y_4$. For $S = \{3, 4\}$, $i = 1$ and $j = 2$, Lemma 4.11 gives

$$Y_{13} Y_{234} = \frac{Y_{1234} Y_{34} + (Y_{34} + Y_3 + Y_4 + 1)(Y_{34} + Y_4)}{Y_4}.$$

Lemma 4.13. *Let $i, k \notin S$, and j be arbitrary such that $i \neq k$ and $j \neq k$. Then*

$$P_{Sk}^{i,j} Y_S = \begin{cases} P_S^{i,k} P_S^{k,j} + P_S^{i,j} Y_{Sk} & \text{if } i \neq j \\ P_S^{i,j} Y_{Sk} & \text{if } i = j. \end{cases}$$

Proof. When $i = j$, the claim follows from $P_{Sk}^{i,j} = Y_{Sk}$ and $P_S^{i,j} = Y_S$. When $i \neq j$ and $j \notin S$, this is a consequence of (4.1) and Lemma 4.10 applied to the matrix $\mathfrak{Y}_{Sik, Sjk}$. One easily verifies that the sign works out correctly in all cases of relative positions of i, j and k . For $j \in S$, we can argue as follows: create a new “virtual” vertex j' outside S , with a single edge from j to j' . Then the claimed identity is the same as that for i, k, j' . \square

4.4. Algebraic dependencies among the Y variables. Lemma 4.11 has the following consequence.

Proposition 4.14. *Let \mathcal{S} be any maximal nested collection with support $S \subset [n]$. Any Y_I for $I \subseteq S$ is a rational function (with integer coefficients) in the Y_{S_i} , $S_i \in \mathcal{S}$.*

Proof. The proof is by induction on size of S . The claim is clearly true for $|S| = 1$. Assume we have verified the claim for all the smaller sizes than the current one. It is clearly enough to consider the case when S is connected, otherwise the claim follows recursively from knowing it for all connected components of S . If $I = S$ the claim is clearly true, so we assume that $I \subsetneq S$.

We first argue that without loss of generality we can assume that I does not contain the maximal element j of S in \mathcal{S} . Indeed, pick any element $k \in S - S^{\max}$ which is not

in I . Let $\mathcal{S}' = \mu_k(\mathcal{S})$ be the maximal nested collection supported on S obtained by exchanging Y_{S_k} for $Y_{(S_{k^+} - \{k\}) \oplus k^+}$. Apply Lemma 4.11 to the product $Y_{S_k} Y_{(S_{k^+} - \{k\}) \oplus k^+}$. (In the notation of Lemma 4.11, our k and k^+ is i and j respectively.) The right hand side of the exchange involves $Y_{S_{k^+}}$, which is one of the Y_{S_i} , $S_i \in \mathcal{S}$. All other Y -s it involves are supported on $S_{k^+} - k^+$, and thus by the induction assumption are rational functions of Y_{S_i} , $S_i \in \mathcal{S}^{k^+} \subset \mathcal{S}$. We conclude that $Y_{(S_{k^+} - \{k\}) \oplus k^+}$ is a rational function in $\{Y_{S_i} \mid S_i \in \mathcal{S}\}$. Thus it suffices to show Y_I is a rational function of $\{Y_{S_i} \mid S_i \in \mathcal{S}'\}$. In \mathcal{S}' the element k is one step closer to being maximal. Proceeding in this manner, we shall arrive at a maximal nested collection on S where $k \notin I$ is maximal.

Now, since $i \notin I$, we have $Y_I = \prod_j Y_{I \cap S_j}$, where the product is over j maximal in \mathcal{S}^i . It is possible this product has just one term, but regardless of that we can now apply the induction assumption to conclude that Y_I is rational function of Y_{S_i} , $S_i \in \mathcal{S}^i$. \square

Example 4.15. Let $S = \{1, 2, 3\}$ in the graph of Example 1.2, and take $\mathcal{S} = \{3\}, \{2, 3\}, \{1, 2, 3\}$. Then the expression for Y_1 in terms of this maximal nested collection is

$$Y_1 = \frac{1 + Y_3^2 + Y_{23} + Y_3(2 + Y_{123})}{Y_3 Y_{23}}.$$

5. GRAPH LP ALGEBRAS

5.1. The LP algebra with linear initial seed. Fix Γ a directed, multiplicity-free, loopless graph on $[n]$. We shall define a LP algebra with coefficient ring $R = \mathbb{Z}[A_1, A_2, \dots, A_n]$, and initial cluster variables X_1, X_2, \dots, X_n . In fact the choice of the coefficient ring R is not critical; the key property we need is that the A_i are algebraically independent.

Let $\mathcal{S} \in \mathfrak{C}$ be a maximal nested collection supported on the set $S \subseteq [n]$. We now define a seed $t_{\mathcal{S}} = \{(X_i^{\mathcal{S}}, F_i^{\mathcal{S}})\}$. The cluster variables are defined by

$$(5.1) \quad X_i^{\mathcal{S}} = \begin{cases} X_i & \text{for } i \notin S \\ Y_{S_i} & \text{for } i \in S. \end{cases}$$

The indices, or cluster variables $X_i^{\mathcal{S}}$ for $i \notin S$ are called *external*, and for $i \in S$ are called *internal* or *activated*. The exchange Laurent polynomials are defined by

$$(5.2) \quad \hat{F}_i^{\mathcal{S}} = \begin{cases} \frac{\sum_{j \notin S_i} P_S^{i,j} X_j + \sum_{j \in S_i} P_S^{i,j} A_j}{Y_{S \ominus i}} & \text{for } i \notin S \text{ or } i \in S^{\max} \\ \frac{Y_{Rij} Y_R + P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{R \ominus j}} & \text{for } i \in S - S^{\max}, j = i^+, \text{ and } R = S_j - \{i, j\}. \end{cases}$$

As is discussed in [LP], the exchange Laurent polynomials \hat{F}_i uniquely determine the exchange polynomials F_i . Note that the expression for \hat{F}_i given by (5.2) is not in terms of cluster variables of $t_{\mathcal{S}}$, but in terms of variables in many different seeds. In the course of our proof of Theorem 5.1, it will be established that $\hat{F}_i^{\mathcal{S}}$ do give legitimate exchange Laurent polynomials when written in terms of the cluster variables of $t_{\mathcal{S}}$.

Suppose $\mathcal{S} = \emptyset$, and thus $S = \emptyset$. Then we have

$$P_{\emptyset}^{i,j} = \begin{cases} 1 & \text{there is an edge } i \rightarrow j \text{ in } \Gamma \text{ or } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus the seed t_\emptyset consists of the cluster variables X_i , and the linear exchange polynomials (which are also the exchange Laurent polynomials \hat{F}_i)

$$F_i = A_i + \sum_{i \rightarrow j} X_j.$$

We call t_\emptyset the *initial seed*. Our main theorem is the following:

Theorem 5.1. *There is a normalized LP algebra $\mathcal{A} = \mathcal{A}_\Gamma$ with seeds exactly the $\{t_\mathcal{S} \mid \mathcal{S} \in \mathfrak{C}\}$, and mutations given by $\mu_i(t_\mathcal{S}) = t_{\mu_i(\mathcal{S})}$.*

Thus \mathcal{A}_Γ has

- (1) clusters $t_\mathcal{S}$ labeled by maximal nested collections $\mathcal{S} \in \mathfrak{C}$.
- (2) cluster variables X_i and Y_S for strongly connected components S of Γ .
- (3) exchange graph equal to the exchange graph on maximal nested collections \mathfrak{C} of Section 3.3.

The proof of Theorem 5.1 will occupy the remainder of this section. We now outline the strategy of the proof. In Section 5.3 we will describe the seeds $t_{\vec{S}}$ that can be obtained from the initial seed t_\emptyset by mutating some sequence $\vec{S} = (s_1, s_2, \dots, s_r) \subset [n]$, one at a time, without mutating any $i \in [n]$ twice. These correspond to the activations of nested collections discussed in Section 3.2. We first describe only the exchange Laurent polynomials of these seeds for external (not-yet activated) vertices which is enough to carry through the inductive calculation, and show that $t_{\vec{S}}$ depends only on \mathcal{S} and not the order of activation \vec{S} . In Section 5.4, we then compute the exchange Laurent polynomials for internal vertices, thus establishing that $t_\mathcal{S} = t_{\vec{S}}$. Finally, in Section 5.5, we shall show that $\mu_i(t_\mathcal{S}) = t_{\mu_i(\mathcal{S})}$ for activated $i \in S - S^{\max}$.

Example 5.2. For the graph in Example 1.2, take $\mathcal{S} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$. We have $t_\mathcal{S} = \mu_3(\mu_2(\mu_1(t_\emptyset)))$. Computing these mutations (and making choices of units so that all exchange polynomials are positive) we obtain that the seed $t_\mathcal{S}$ is given by

$$\begin{aligned} (Y_1, F_1 &= 1 + Y_{12}) \\ (Y_{12}, F_2 &= 1 + Y_1^2 + Y_1(2 + Y_{123})) \\ (Y_{123}, F_3 &= X_4(1 + Y_1)(1 + Y_{12}) + A_1(1 + Y_1 + Y_{12}) + A_2Y_1(1 + Y_1) + A_3Y_1Y_{12}) \\ (X_4, F_4 &= A_1(1 + Y_{12} + Y_1^2 + Y_1(2 + Y_{123}) + Y_1Y_{12}) + \\ &\quad A_2(Y_1^3 + Y_1^2(2 + Y_{123}) + Y_1) + A_3(Y_1^2Y_{12} + Y_1Y_{12}) + A_4Y_1Y_{12}Y_{123}). \end{aligned}$$

Here, all exchange polynomials have been written in terms of the cluster variables $\{Y_1, Y_{12}, Y_{123}, X_4\}$ of $t_\mathcal{S}$. One checks that $\hat{F}_3 = F_3/Y_1$, and $\hat{F}_4 = F_4/(Y_1Y_{12})$, and thus:

$$\begin{aligned} \hat{F}_3 &= \frac{X_4(1 + Y_1)(1 + Y_{12}) + A_1(1 + Y_1 + Y_{12}) + A_2Y_1(1 + Y_1) + A_3Y_1Y_{12}}{Y_1} \\ &= X_4(Y_{12} + Y_2 + 1) + A_1(1 + Y_2) + A_2(1 + Y_1) + A_3Y_{12} \end{aligned}$$

and

$$\begin{aligned}\hat{F}_4 &= \frac{1}{Y_1 Y_{12}} (A_1(1 + Y_{12} + Y_1^2 + Y_1(2 + Y_{123}) + Y_1 Y_{12}) \\ &\quad + A_2(Y_1^3 + Y_1^2(2 + Y_{123}) + Y_1) + A_3(Y_1^2 Y_{12} + Y_1 Y_{12}) + A_4 Y_1 Y_{12} Y_{123}) \\ &= A_1(Y_3 + 1) + A_2 Y_{13} + A_3(Y_1 + 1) + A_4 Y_{123}\end{aligned}$$

agreeing with (5.2). For example, the coefficient $Y_{12} + Y_2 + 1$ of X_4 in \hat{F}_3 is a summation over the following three paths from 3 to 4 via $S = \{1, 2\}$:

$$3 \longrightarrow 4, \quad 3 \longrightarrow 1 \longrightarrow 4, \quad 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 4.$$

5.2. Rigidity and irreducibility of exchange Laurent polynomials. In this subsection we will make some general observations concerning the polynomials defined in (5.2). Let $\mathcal{L}(t)$ denote the Laurent polynomial ring of the seed t . If t has cluster variables x_1, x_2, \dots, x_n , then $\mathcal{L}(t) = R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Lemma 5.3. *The cluster variables X_i^S of t_S are algebraically independent.*

Proof. Follows easily from Proposition 4.14. \square

We note that all the denominators of \hat{F}_i^S defined in (5.2) are monomials in $\mathcal{L}(t_S)$. For $i \notin S$ or $i \in S^{\max}$, we have that $S \ominus i$ is the union of some of the strongly connected components of S . For $i \in S - S^{\max}$, both $S \ominus i$ and $S \ominus j$ are unions of some of the strongly connected components of R . The strongly connected components of R are exactly the $S_k \in \mathcal{S}$ such that $k^+ = i$ or $k^+ = j$.

Proposition 5.4. *Let $\mathcal{S} \in \mathfrak{C}$ be a maximal nested collection, and suppose it is known that*

$$(5.3) \quad \{Y_I \mid I \subset S\} \subset \mathcal{L}(t_S).$$

Then the exchange polynomials \hat{F}_i^S of (5.2) are irreducible in $\mathcal{L}(t_S)$.

Proof. Suppose first $i \in S^{\max}$ or $i \notin S$. The denominator $Y_{S \ominus i}$ is a monomial in the cluster variables of $\mathcal{L}(t_S)$, and each $P_S^{i,j}$ is a polynomial in the $\{Y_I \mid I \subset S\}$, so it follows from (5.3) that we have $\hat{F}_i^S \propto M A_i + N$ for a monomial M in the $\{X_k^S \mid k \in [n]\}$ and a polynomial $N \in R[X_i^S]$ which does not involve A_i . It follows easily that \hat{F}_i^S is irreducible in $\mathcal{L}(t_S)$.

Now if $i \in S - S^{\max}$, then again the denominator $Y_{R \ominus i} Y_{R \ominus j}$ is checked to be a unit in $\mathcal{L}(t_S)$. Now we apply the same argument as before but with the cluster variable Y_{Rij} instead of A_i . Note that by (5.3) and Proposition 4.14, Y_I for $I \subset R$ will be a Laurent polynomial not involving Y_{Rij} . \square

Equation (5.3) will follow from Corollary 5.7 below.

It turns out one does not have to compute the exchange Laurent polynomials explicitly, but only up to a monomial factor in the cluster variables of the corresponding cluster. This will simplify the subsequent calculations significantly. Let $\mathcal{A}(t_\emptyset)$ denote the LP algebra generated by t_\emptyset . The LP algebra $\mathcal{A}(t_\emptyset)$ is only defined up to choices of a unit $\pm 1 \in R$ for all the mutations; we fix these choices by insisting that all cluster variables of $\mathcal{A}(t_\emptyset)$ are Laurent polynomials in X_1, X_2, \dots, X_n with positive coefficients. That such a choice can be made will follow from our constructions. It is also clear that this choice is compatible with Theorem 2.4.

Proposition 5.5. *Let $\mathcal{S} \in \mathfrak{C}$, and assume it is known that $\{Y_I \mid I \subset S\}$ are all cluster variables of $A(t_\emptyset)$. Suppose $t = \{(X_i^{\mathcal{S}}, F_i)\}$ is a seed of $A(t_\emptyset)$ with the same cluster variables $X_i^{\mathcal{S}}$ as $t_{\mathcal{S}}$ but possibly different exchange polynomials. Suppose also that the exchange Laurent polynomials \hat{F}_i for $i \notin S$ or $i \in S^{\max}$ are all known to be equal to $\hat{F}_i^{\mathcal{S}}$ of (5.2) up to a monomial in the $\{X_r^{\mathcal{S}} \mid r \in [n]\}$. Then $\hat{F}_i = \hat{F}_i^{\mathcal{S}}$ for all $i \notin S$ or $i \in S^{\max}$. Furthermore if for any $i \in S - S^{\max}$ we have that \hat{F}_i is equal to $\hat{F}_i^{\mathcal{S}}$ up to a monomial factor, then $\hat{F}_i = \hat{F}_i^{\mathcal{S}}$ as well.*

Proof. First suppose that $i \notin S$ or $i \in S^{\max}$. We need to argue that \hat{F}_i is equal to

$$\hat{F}_i^{\mathcal{S}} = \frac{\sum_{j \notin Si} P_S^{i,j} X_j + \sum_{j \in Si} P_S^{i,j} A_j}{Y_{S \ominus i}}.$$

Applying Lemma 4.8 and Lemma 2.3, we conclude \hat{F}_i is obtained from $\hat{F}_i^{\mathcal{S}}$ by possibly dividing by *more* of the Y variables in t . The X -variables cannot occur in \hat{F}_i/F_i since for $k \notin S$, \hat{F}_k depends on A_k which does not occur in \hat{F}_i .

On the other hand, if we have $\hat{F}_i = \hat{F}_i^{\mathcal{S}} M$ where M is a (negatively-powered) monomial in the Y variables, then the cluster variable $X'_i = \hat{F}_i/X_i$ in $\mu_i(t)$ would be equal to

$$\frac{\hat{F}_i}{X_i} = M \frac{\hat{F}_i^{\mathcal{S}}}{X_i} = M Y_{S \oplus i},$$

where the last equality is by Lemma 4.7. This is impossible by Lemma 4.4 and the fact that the resulting cluster variable has to be a Laurent polynomial in the t_\emptyset cluster.

Now suppose $i \in S - S^{\max}$, and set $j = i^+$ and $R = S_j - \{i, j\}$. Then

$$\hat{F}_i^{\mathcal{S}} = \frac{Y_{Rij} Y_R + P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{R \ominus j}}.$$

Just as before, none of the X_k variables can be in the denominator \hat{F}_i/F_i . The rest of the argument is the same, with Lemma 4.11 and $Y_{R \oplus j}$ instead of Lemma 4.7 and $Y_{S \oplus i}$. \square

5.3. Clusters reachable from the initial seed via an activation sequence. Let $\vec{S} = (s_1, s_2, \dots, s_r)$ be an activation sequence giving a maximal nested collection \mathcal{S} with support S . Denote by $t_{\vec{S}}$ the seed $\mu_{s_r} \circ \dots \circ \mu_{s_1}(t_\emptyset)$ of $\mathcal{A}(t_\emptyset)$.

Proposition 5.6. *In $t_{\vec{S}}$*

- (1) *the cluster variables are those of $t_{\mathcal{S}}$ as defined in (5.1)*
- (2) *for all $i \notin S$ or $i \in S^{\max}$ the exchange Laurent polynomials \hat{F}_i agree with the definition (5.2) of $\hat{F}_i^{\mathcal{S}}$ for $t_{\mathcal{S}}$.*
- (3) *the seed $t_{\vec{S}}$ depends only on \mathcal{S} , and not on the actual activation sequence \vec{S} .*

Corollary 5.7. *For each strongly connected subset $T \subset [n]$ and each $\mathcal{S} \in \mathfrak{C}$ we have $Y_T \in \mathcal{L}(t_{\mathcal{S}})$.*

Proof. Follows immediately from Proposition 5.6(1) and Theorem 2.2. \square

Proof of Proposition 5.6. We prove the three claims by simultaneous induction on the size of the support S . The base case $|S| = 0$ is clear.

Assume the result has been shown for $\vec{S}' = (s_1, s_2, \dots, s_{r-1})$ and we now want to verify it for $\vec{S} = (s_1, s_2, \dots, s_{r-1}, k)$ where $k \notin S'$. Write \mathcal{S}' for the maximal nested collection with activation sequence \vec{S}' , and similarly for \mathcal{S} . Note that $S = S' \cup \{k\}$. We shall write \hat{F}_i (resp. \hat{F}'_i) for the Laurent exchange polynomials of the seed $t = t_{\vec{S}}$ (resp. $t' = t_{\vec{S}'}$). By definition and the inductive hypothesis, the cluster variable X_k in $t_{\vec{S}'}$ is exchanged for

$$\hat{F}'_i / X_k = \frac{\sum_{j \notin S_k} P_S^{k,j} X_j + \sum_{j \in S_k} P_S^{k,j} A_j}{Y_{S \oplus k} X_k} = Y_{S \oplus k}$$

by Lemma 4.7. But $\mathcal{S} = \mathcal{S}' \cup \{S \oplus k\}$, so we have verified claim (1).

We now consider claim (2). We first make the general observation that by Theorem 2.2, each \hat{F}'_i of (5.2) can be seen to lie in $\mathcal{L}(t')$, since each Y_T for $T \subset S'$ is known to be a cluster variable of $\mathcal{A}(t_\emptyset)$.

We know that $\hat{F}_k = \hat{F}'_k$ and this easily agrees with (5.2).

Now suppose $i \in (S')^{\max} \cap S^{\max}$ is maximal in both S' and in S . Thus i and k are not in the same strongly connected component of S , so either there is no path from i to k through S , or there is no path from k to i through S .

In the first case, we see that the cluster variable X_k does not appear in \hat{F}'_i . Therefore $F_i = F'_i$, so $\hat{F}_i = \hat{F}'_i$ up to a monomial in $\mathcal{L}(t)$. But we note that $\hat{F}'_i = \hat{F}_i^{S'}$ is actually equal to the predicted \hat{F}_i^S for the following reason. Since we cannot get from i to $S \oplus k$ via S (resp. S'), each of the $P_S^{i,\ell}$ (resp. $P_{S'}^{i,\ell}$) actually factors, producing $Y_{S \oplus k}$ (resp. $Y_{(S \oplus k) - \{k\}}$) factors. This same factor appears in the denominator of \hat{F}_i^S (resp. $\hat{F}_i^{S'}$) which is $Y_{S \oplus i}$ (resp. $Y_{S' \oplus i}$). After these factors are canceled, we see that $\hat{F}_i^S = \hat{F}_i^{S'}$. It follows from Proposition 5.5 that $\hat{F}_i = \hat{F}_i^S$.

In the second case, consider the seed $t_{\vec{S}''}$ with $S'' = S' - \{i\}$ and such that activation of i gives t' . This seed exists by the inductive hypothesis (3). We have two cluster variables in $t_{\vec{S}''}$, X_i and X_k , such that X_i does not appear in \hat{F}_k (and thus also not in F_k). Then by Theorem 2.4 we know the two mutations commute, and \hat{F}_i after activating i and then k is the same as the exchange Laurent polynomial $\hat{F}_i^{S''}$ in the seed $t''' = \mu_k(t_{\vec{S}''})$ after just activating k . But t''' falls inside our inductive hypothesis. It remains to note that $P_{S - \{i\}}^{i,\ell} = P_{S'}^{i,\ell}$, and thus $\hat{F}_i = \hat{F}_i^{S''} = \hat{F}_i^{S'''} = \hat{F}_i^S$, establishing (2) in this case.

Consider now the case when $i \notin S$ is external. The case where there is no path from i to k through S is treated the same way as for $i \in (S')^{\max} \cap S^{\max}$. Assume a path from i to k through S exists. By the induction assumption, we have

$$\hat{F}'_i = \frac{\sum_{\ell \notin Si} P_S^{i,\ell} X_\ell + \sum_{\ell \in Si} P_S^{i,\ell} A_\ell}{Y_{S \oplus i}}$$

and

$$\hat{F}'_k|_{X_i=0} = \frac{\sum_{\ell \notin Sik} P_S^{k,\ell} X_\ell + \sum_{\ell \in Sik} P_S^{k,\ell} A_\ell}{Y_{S \oplus k}}.$$

By Remark 2.1, we can calculate \hat{F}_i by substituting into \hat{F}'_i .

Substituting $\hat{F}'_k|_{X_i=0}/Y_{S\oplus k}$ for X_k in \hat{F}'_i gives

$$\frac{1}{Y_{S\oplus k}} \sum_{\ell \neq i} \left(\frac{P_S^{k,\ell}}{Y_{S\oplus k}} \cdot \frac{P_S^{i,k}}{Y_{S\oplus i}} + \frac{P_S^{i,\ell}}{Y_{S\oplus i}} \cdot Y_{S\oplus k} \right) \tilde{X}_\ell + \frac{P_S^{i,i}}{Y_{S\oplus i}} \cdot Y_{S\oplus k} X_i = \frac{1}{Y_{S\oplus k}} \sum_{\ell} P_S^{i,\ell} \frac{Y_S}{Y_{S\oplus k} Y_{S\oplus i}} \tilde{X}_\ell$$

where \tilde{X}_ℓ is equal to X_ℓ or A_ℓ and we have applied Lemma 4.13, noting that $Y_{S\oplus k} Y_{S\oplus i} = Y_{S\oplus k}$. The factor $\frac{Y_S}{Y_{S\oplus k} Y_{S\oplus i}}$ does not depend on j so together with Proposition 5.4 we have that

$$\hat{F}_i \propto \sum_{j \notin S \cup i} P_{S\oplus k}^{i,j} X_j + \sum_{j \in S \cup i} P_{S\oplus k}^{i,j} A_j.$$

The proof of (2) is then completed by applying Proposition 5.5.

Finally, to establish (3), let $\vec{S}'' = (s_1, s_2, \dots, s_{r-2})$ and suppose s_{r-1} and s_{r-2} are exchangeable in $\vec{S} = (s_1, s_2, \dots, s_{r-2}, s_{r-1}, s_r)$. Then s_{r-1} and s_r are not strongly connected via S'' , so it follows that one of $\hat{F}_{s_{r-1}}$ and \hat{F}_{s_r} does not involve the other's cluster variable ($X_{s_{r-1}}$ or X_{s_r}). By Theorem 2.4 we have $\mu_{s_r} \mu_{s_{r-1}}(t_{\vec{S}''}) = \mu_{s_{r-1}} \mu_{s_r}(t_{\vec{S}''})$. This together with the inductive hypothesis completes the proof of (3). \square

5.4. The internal exchange polynomials.

Proposition 5.8. *The exchange Laurent polynomials \hat{F}_i of $t_{\vec{S}}$ for $i \in S - S^{\max}$ agree with those of t_S defined in (5.2).*

Together with Proposition 5.6 we thus have $t_{\vec{S}} = t_S$.

Proof. Fix $i \in S - S^{\max}$ and set $j = i^+$ and $R = S_j - \{i, j\}$. Note that $S_i = R \oplus i$ and $S_j = R \cup j = R \oplus j$.

It suffices to show that the statement is true when j is the unique maximal element of S , so that $S_j = S$. To see this, observe by Lemma 3.6 and Proposition 5.6 that $t_{\vec{S}}$ can be created by first activating all the elements in S_j (in some order), and then activating elements outside of S_j in some order. But by Proposition 4.14 the expression $\frac{Y_{R \cup j} Y_R + P_R^{i,j} P_R^{j,i}}{Y_{R \cup i} Y_{R \cup j}}$ can be expressed purely in terms of the Y variables inside S_j , and does not depend on X variables. Then Theorem 2.4 states that activation of variables outside S_j commutes with mutation of Y_{S_i} , and in particular activation of variables outside S_j preserves the value of \hat{F}_i . We shall thus assume from now on that $S = S_j$.

Let $t' = t_{\vec{S}'}$ be the cluster with i maximal such that mutating at j gives $t = t_{\vec{S}}$, so we have $S' \cup \{j\} = S = S_j$. Denote by F'_i the exchange polynomials of the cluster t' . By Proposition 5.6,

$$\hat{F}'_i = \frac{\sum_{\ell \notin R \cup i} P_R^{i,\ell} X_\ell + \sum_{\ell \in R \cup i} P_R^{i,\ell} A_\ell}{Y_{R \cup i}}$$

and

$$\hat{F}'_j = \frac{\sum_{\ell \notin R \cup j} P_{R \cup j}^{j,\ell} X_\ell + \sum_{\ell \in R \cup j} P_{R \cup j}^{j,\ell} A_\ell}{Y_{R \cup j}}.$$

Use the notation

$$(5.4) \quad \tilde{X}_\ell = \begin{cases} X_\ell & \text{if } \ell \notin R \cup i \cup j; \\ A_\ell & \text{otherwise.} \end{cases}$$

Then noting that $Ri \ominus j = \emptyset$, we have

$$\hat{F}'_j = \frac{\sum_{\ell} P_{Ri}^{j,\ell} \tilde{X}_{\ell}}{Y_{(Ri) \ominus j}} = \sum_{\ell} P_{Ri}^{j,\ell} \tilde{X}_{\ell}.$$

We can use Lemma 4.13 to obtain

$$\hat{F}'_j = \frac{\sum_{\ell} P_R^{j,\ell} Y_{Ri} \tilde{X}_{\ell} + \sum_{\ell \neq j} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R}.$$

Applying Proposition 4.14 to this expression, we can write it as a rational function in the cluster variables of $t_{S'}$. The variable $Y_{S_i} = Y_{Ri}$ does not appear in $P_R^{j,\ell}$, $P_R^{j,i}$, or $P_R^{i,\ell}$ so we see that setting $Y_{S_i} = 0$ in \hat{F}'_j gives

$$\frac{\sum_{\ell \neq j} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R}.$$

Substituting for X_j in \hat{F}'_i gives

$$(\hat{F}'_i)_{X_j \leftarrow \frac{\sum_{\ell} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R Y_{S_j}}} = \frac{1}{Y_{R \ominus i}} \left(P_R^{i,j} \frac{\sum_{\ell} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R Y_{S_j}} + \sum_{\ell} P_R^{i,\ell} \tilde{X}_{\ell} \right),$$

where throughout the summation is over $\ell \neq j$. Dividing by the common factor $\sum_{\ell} P_R^{i,\ell} \tilde{X}_{\ell}$ with $\hat{F}'_j|_{Y_{S_i}=0}$ and using Proposition 5.4, we obtain

$$\hat{F}_i \propto Y_{Rij} Y_R + P_R^{i,j} P_R^{j,i}.$$

The claim then follows from Proposition 5.5. □

5.5. Proof of Theorem 5.1. Let t_S be one of the clusters in $\mathcal{A}(t_{\emptyset})$ that have been studied in Propositions 5.6 and 5.8, and $t' = \mu_i(t_S)$ be the cluster obtained by mutation at i , where $i \in S - S^{\max}$. Our aim is to show that $t' = t_{\mu_i(S)}$. Set $S' = \mu_i(S)$.

Proposition 5.8 and Lemma 4.11 imply that the cluster variables of t' and $t_{S'}$ are the same. It remains to check that the exchange polynomials F'_k of t' are also correct, and for that it suffices to check that the \hat{F}'_k 's are correct. Just as in the proof of Proposition 5.8, we may assume that $S = S_{i^+}$, since all further activating mutations commute with the one exchanging Y_{S_i} for $Y_{(S-\{i\}) \oplus i^+}$. Let $j = i^+$ as before, and let $K \subset S$ be the set of k such that $k^+ = i$ or $k^+ = j$. Since \hat{F}_i does not depend on external X variables, its mutation commutes with theirs by Theorem 2.4, and thus does not change the corresponding \hat{F} . The same is true for elements of $r \in S$ different from j and not in K : their exchange polynomial F_r does not depend on Y_{S_i} and also is not equal to F_i (even up to a unit in $\mathcal{L}(t_S)$).

Thus we only need to check that \hat{F}_j and \hat{F}_k for $k \in K$ mutate correctly. Set $R = S_j - \{i, j\}$ as before. Setting $Y_{Rij} = 0$ inside

$$\hat{F}_i = \frac{Y_{Rij} Y_R + P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{R \ominus j}}$$

we obtain

$$\frac{P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{R \ominus j}},$$

since by Proposition 4.14 this expression does not depend on $Y_{Ri,j}$. Plugging

$$Y_{R \oplus i} \longleftarrow \frac{P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{R \ominus j} Y_{R \oplus j}}$$

in

$$\hat{F}_j = \sum_{\ell} P_{Ri}^{j,\ell} \tilde{X}_{\ell} = \frac{\sum_{\ell} P_R^{j,\ell} Y_{Ri} \tilde{X}_{\ell} + \sum_{\ell \neq j} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R}$$

where we are using the notation of (5.4), we get

$$\frac{\sum_{\ell} P_R^{j,\ell} \frac{P_R^{i,j} P_R^{j,i}}{Y_{R \ominus i} Y_{Rj}} Y_{R \ominus i} \tilde{X}_{\ell} + \sum_{\ell} P_R^{j,i} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R} = \frac{\frac{P_R^{i,j} P_R^{j,i}}{Y_{Rj}} \sum_{\ell} P_R^{j,\ell} \tilde{X}_{\ell} + P_R^{j,i} \sum_{\ell} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R}.$$

Canceling out the common factor $P_R^{j,i}$ we get

$$\hat{F}'_j \propto \frac{\frac{P_R^{i,j}}{Y_{Rj}} \sum_{\ell} P_R^{j,\ell} \tilde{X}_{\ell} + \sum_{\ell} P_R^{i,\ell} \tilde{X}_{\ell}}{Y_R} = \frac{\sum_{\ell} P_{Rj}^{i,\ell} \tilde{X}_{\ell}}{Y_{Rj}}$$

by Lemma 4.13 (for the last equality) and Proposition 5.4. We then see that $\hat{F}'_j = \hat{F}_j^{S'}$ by Proposition 5.5.

It remains to check what happens with \hat{F}_k , for $k \in K$. Let $T = S - i - k - j = R - k$. We have

$$(5.5) \quad \hat{F}_i = \frac{Y_{Tijk} Y_{Tk} + P_{Tk}^{i,j} P_{Tk}^{j,i}}{Y_{Tk \ominus i} Y_{Tk \ominus j}}.$$

Case $k^+ = j$: Suppose first that $k^+ = j$, so that k and i are not strongly connected via T , and thus $P_T^{i,k} P_T^{k,i} = 0$. We have

$$(5.6) \quad \hat{F}_k = \frac{Y_{Tikj} Y_{Ti} + P_{Ti}^{k,j} P_{Ti}^{j,k}}{Y_{Ti \ominus k} Y_{Ti \ominus j}}$$

Applying Lemma 4.13 to $P_{Ti}^{k,j}$ and $P_{Ti}^{j,k}$ gives

$$P_{Ti}^{k,j} Y_T = P_T^{k,i} P_T^{i,j} + P_T^{k,j} Y_{Ti} \quad P_{Ti}^{j,k} Y_T = P_T^{j,i} P_T^{i,k} + P_T^{j,k} Y_{Ti}.$$

Using $Y_{Ti} = Y_{T \ominus i} Y_{T \oplus i}$ and $Y_{Ti \ominus k} = Y_{(T \ominus i) \ominus k} Y_{T \oplus i}$ we get

$$(5.7) \quad \hat{F}_k = \frac{Y_T^2 Y_S Y_{T \ominus i} + Y_{T \ominus i} (P_T^{k,i} P_T^{i,j} P_T^{j,k} + P_T^{j,i} P_T^{i,k} P_T^{k,j}) + P_T^{k,j} P_T^{j,k} Y_{T \ominus i} Y_{Ti}}{Y_{(T \ominus i) \ominus k} Y_{Ti \ominus j} Y_T^2}$$

Subcase $P_T^{k,j} P_T^{j,k} = 0$: Thus k and j are not strongly connected via T and in S' , we have $k^+ = i$. In this subcase, Y_{S_i} only occurs in the denominator of \hat{F}_k , so this means F_k does not involve Y_{S_i} . It follows that $F'_k = F_k$. If $t' = t_{S'}$, according to Theorem ?? we should have

$$(5.8) \quad \hat{F}'_k = \frac{Y_T^2 Y_{Tikj} Y_{Tj} + P_{Tj}^{k,i} P_{Tj}^{i,k}}{Y_{Tj \ominus k} Y_{Tj \ominus i}} = \frac{Y_S Y_{T \ominus j} + Y_{T \ominus j} (P_T^{k,i} P_T^{i,j} P_T^{j,k} + P_T^{j,i} P_T^{i,k} P_T^{k,j})}{Y_T^2 Y_{(T \ominus j) \ominus k} Y_{Tj \ominus i}}$$

If i and j are strongly connected via T , then $Tj \ominus i = Ti \ominus j$, and we see that (5.8) and (5.7) differ (multiplicatively) by factors of Y_V for variables V which are present in both t_S and t' . This implies that up to a monomial factor (5.7) is the correct formula for \hat{F}'_k in t' , and we may now apply Proposition 5.5.

If i and j are not strongly connected via T . Then $Y_{Tj \ominus i}$ has a factor of the new cluster variable $Y_{T \oplus j}$ in t' while $Y_{Ti \ominus j}$ has a factor of the old cluster variable $Y_{T \oplus i}$ in t_S . These factors are of course not present in the corresponding non-Laurent exchange polynomials F_k . In any case, we still conclude that (5.7) is the correct formula for \hat{F}'_k in t' .

Subcase $P_T^{k,j} P_T^{j,k} \neq 0$: Thus j and k are strongly connected via T , and in \mathcal{S}' , we have $k^+ = j$. In the denominator of (5.5), we have $Y_{Tk \ominus i} = Y_{T \oplus k} Y_{(Tk \ominus k) \ominus i}$ but $Y_{Tk \ominus j}$ does not involve the variable $Y_{T \oplus k}$. Applying Lemma 4.13, we get

$$\hat{F}_i = \frac{Y_T^2 Y_{Tij} Y_{T \ominus k} + Y_{T \ominus k} (P_T^{i,k} P_T^{k,j} P_T^{j,i} + P_T^{i,j} P_T^{j,k} P_T^{k,i}) + P_T^{i,j} P_T^{j,i} Y_{Tk} Y_{T \ominus k}}{Y_{(Tk \ominus k) \ominus i} Y_{Tk \ominus j} Y_T^2}$$

None of the $P_T^{a,b}$ involve $Y_{T \oplus k}$ when expressed as an element of $\mathcal{L}(t_S)$, so substituting $Y_{T \oplus k} = 0$, we obtain

$$\hat{F}_i|_{Y_{T \oplus k}=0}/Y_{Tk \oplus j} = \frac{Y_T^2 Y_{Tij} Y_{T \ominus k} + Y_{T \ominus k} (P_T^{i,k} P_T^{k,j} P_T^{j,i} + P_T^{i,j} P_T^{j,k} P_T^{k,i})}{Y_{(Tk \ominus k) \ominus i} Y_{Tk \ominus j} Y_T^2 Y_{Tk \oplus j}}$$

Substituting this last expression for $Y_{T \oplus i} = Y_{S_i}$ in (5.7) (again we use the fact that the expression in (5.7) is linear in Y_{S_i}), we obtain, up to an overall monomial factor

$$\begin{aligned} & Y_T^4 Y_S Y_{Tk} Y_{(Tk \ominus k) \ominus i} + Z Y_T^2 Y_{Tk} Y_{(Tk \ominus k) \ominus i} + P_T^{k,j} P_T^{j,k} Y_S Y_T^3 Y_{(Tk \ominus k) \ominus i} + P_T^{k,j} P_T^{j,k} Y_T Y_{(Tk \ominus k) \ominus i} Z \\ &= (Y_{Tkj} Y_T + P_T^{k,j} P_T^{j,k}) (Y_T^2 Y_S + Z) Y_T Y_{(Tk \ominus k) \ominus i} \end{aligned}$$

where $Z = P_T^{i,k} P_T^{k,j} P_T^{j,i} + P_T^{i,j} P_T^{j,k} P_T^{k,i}$, and we have used the equalities $Y_T Y_{(Tk \ominus k) \ominus i} = Y_{T \ominus i} Y_{T \ominus k}$ and $Y_{Tk} = Y_{Tk \ominus j} Y_{Tk \oplus j}$. The factor $(Y_T^2 Y_S + Z)$ is a common factor with $\hat{F}_i|_{Y_{T \oplus k}=0}$, so we conclude from Proposition 5.4 that

$$\hat{F}'_k \propto Y_{Tkj} Y_T + P_T^{k,j} P_T^{j,k}.$$

The claimed statement for \hat{F}'_k then follows from Proposition 5.5.

Case $k^+ = i$: In this case we have

$$(5.9) \quad \hat{F}_k = \frac{Y_{Tik} Y_T + P_T^{i,k} P_T^{k,i}}{Y_{T \ominus i} Y_{T \ominus k}}.$$

Subcase $P_T^{j,k} P_T^{k,j} = 0$: In this subcase, we have $k^+ = i$ in $t_{S'}$.

Applying Lemma 4.13 to (5.5), and using $Y_{Tk \ominus j} = Y_{(Tk \ominus j) \ominus k} Y_{T \oplus k}$ we get

$$(5.10) \quad \hat{F}_i = \frac{Y_T^2 Y_{Tij} Y_{T \ominus k} + Y_{T \ominus k} (P_T^{i,k} P_T^{k,j} P_T^{j,i} + P_T^{i,j} P_T^{j,k} P_T^{k,i}) + P_T^{i,j} P_T^{j,i} Y_{Tk} Y_{T \ominus k}}{Y_{(Tk \ominus j) \ominus k} Y_{Tk \ominus i} Y_T^2}$$

Plugging $Y_{T \oplus k} = 0$ into (5.10) we get

$$\hat{F}_i|_{Y_{T \oplus k}=0} = \frac{Y_T^2 Y_{Tij} Y_{T \ominus k} + Y_{T \ominus k} Z}{Y_T^2 Y_{Tk \ominus i} Y_{(T \ominus k) \ominus j}}$$

where $Z = P_T^{i,k} P_T^{k,j} P_T^{j,i} + P_T^{i,j} P_T^{j,k} P_T^{k,i}$. Now substituting

$$\frac{\hat{F}_i|_{Y_{T \oplus k}=0}}{Y_{T \oplus j}} = \frac{Y_T^2 Y_{Tij} Y_{T \oplus k} + Y_{T \oplus k} Z}{Y_{T \oplus j} Y_T^2 Y_{T \oplus i} Y_{(T \oplus k) \oplus j}}$$

for $Y_{S_i} = Y_{T \oplus i}$ in \hat{F}_k , we get

$$\frac{Y_T^2 Y_{Tij} Y_{T \oplus k} + Y_{T \oplus k} Z}{Y_T^2 Y_{T \oplus i} Y_{(T \oplus k) \oplus j} Y_{T \oplus j}} Y_{T \oplus i} Y_T + P_T^{i,k} P_T^{k,i}$$

up to a monomial factor in $\mathcal{L}(t')$. Now $Y_{(T \oplus k) \oplus j} Y_{T \oplus j} = Y_{T \oplus k}$ and $Y_{T \oplus j} Y_T = Y_{Tj} Y_{T \oplus k}$ so we have

$$\begin{aligned} \hat{F}'_k &\propto Y_T^2 Y_{Tij} Y_{T \oplus k} + Y_{T \oplus k} Z + Y_{T \oplus k} P_T^{i,k} P_T^{k,i} Y_T \\ &\propto Y_T^2 Y_{Tij} Y_{Tj} + Y_{Tj} Z + P_T^{i,k} P_T^{k,i} Y_{Tj}^2 \\ &= Y_T^2 (Y_{Tij} Y_{Tj} + P_T^{i,k} P_T^{k,i}) \end{aligned}$$

where in the last step we have used Lemma 4.13. But this last expression, is up to a monomial in $\mathcal{L}(t')$ equal to the expression for \hat{F}'_k predicted by (5.2). The claimed statement for \hat{F}'_k then follows from Proposition 5.5.

Subcase $P_T^{j,k} P_T^{k,i} \neq 0$ In this case, we have $k^+ = j$ in $t_{S'}$. In (5.5) neither factor in the denominator is divisible by $Y_{T \oplus k}$, so substituting $Y_{T \oplus k} = 0$ into \hat{F}_i , the term $Y_{Tij} Y_{T \oplus k}$ is killed since $Y_{T \oplus k} = Y_{T \oplus i} Y_{T \oplus j}$. For the other term $P_T^{i,j} P_T^{j,i}$ we apply Lemma 4.13 and obtain

$$\hat{F}_i|_{Y_{T \oplus k}=0} = \frac{P_T^{i,k} P_T^{k,j} P_T^{j,i} P_T^{k,i}}{Y_T^2 Y_{T \oplus i} Y_{T \oplus j}}.$$

Substituting $\frac{P_T^{i,k} P_T^{k,j} P_T^{j,i} P_T^{k,i}}{Y_T^2 Y_{T \oplus i} Y_{T \oplus j} Y_{T \oplus k}}$ for $Y_{T \oplus i}$ into (5.9) (using $Y_{Tik} = Y_{T \oplus i} Y_{T \oplus k}$), we obtain up to a monomial factor

$$\frac{P_T^{i,k} P_T^{k,j} P_T^{j,i} P_T^{k,i}}{Y_T^2 Y_{T \oplus i} Y_{T \oplus j}} Y_{T \oplus i} Y_T + P_T^{i,k} P_T^{k,i}.$$

Killing the common factor $P_T^{i,k} P_T^{k,i}$ with $\hat{F}_i|_{Y_{T \oplus k}=0}$ and bringing to a common denominator, we observe that this is up to a monomial factor the exchange polynomial \hat{F}'_k in the cluster $t_{S'}$ given by (5.2). The required result then follows from Proposition 5.5.

6. EXAMPLES

6.1. Paths. Consider the path P_n on $[n]$ that consists of edges $i \rightarrow i+1$ for $i \in [n-1]$ and $i \rightarrow i-1$ for $i \in [2, n]$. The LP algebra \mathcal{A}_{P_n} is the *path LP algebra*. Let \mathcal{A}'_{P_n} be the rank $n-1$ LP algebra defined in Section 1.6. The seeds of \mathcal{A}'_{P_n} are exactly the seeds of \mathcal{A}_{P_n} that contain $Y_{[n]}$, and with $Y_{[n]}$ removed.

The strongly connected components P_n are exactly the intervals $\{[a, b] \mid 1 \leq a \leq b \leq n\}$. There is a bijection between maximal nested families of P_n with support $[n]$, and triangulations of an $(n+1)$ -gon. The bijection is obtained by mapping a set $[a, b]$ into the diagonal which cuts vertices a through b away from the rest of the vertices. An example is given in Figure 2. The complex of partial triangulations of the $n+1$ -gon give a simplicial complex dual to the associahedron.

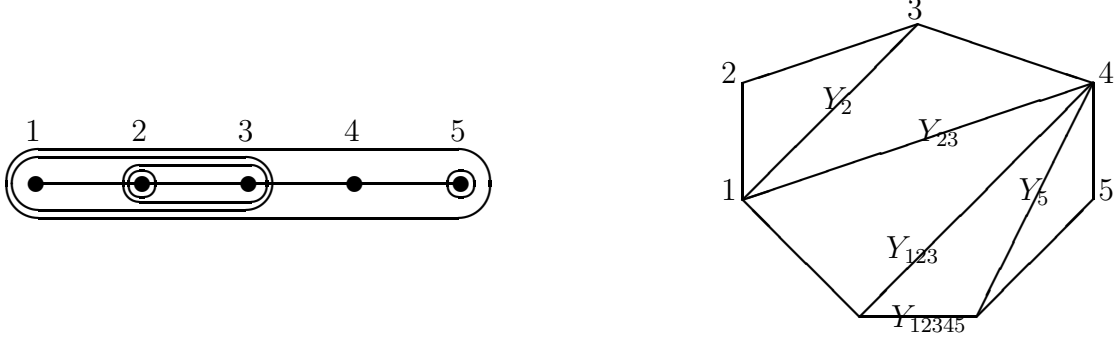


FIGURE 2. The nested collection $\{2, 23, 123, 5, 12345\}$ on the left corresponds to the triangulation on the right.

Theorem 6.1. *Let t_S be a seed of \mathcal{A}'_{P_n} . Suppose $S_i = I \cup \{i\} \cup J \in \mathcal{S}$, where $I, J \in \mathcal{S} \cup \{\emptyset\}$. Then the exchange relation for Y_{S_i} has the form*

$$Y_{IiJ}Y_{JjK} = Y_JY_{IiJjK} + Y_IY_K.$$

where $K \in \mathcal{S} \cup \{\emptyset\}$ is such that $j = i^+$ and $S_j = I \cup \{i\} \cup J \cup \{j\} \cup K$.

Proof. The claim follows immediately from the formula for exchange Laurent polynomials in (5.2). \square

The *Ptolemy cluster algebra* of type A_{n-1} is often modeled by identifying clusters with triangulations of a $[n+1]$ -gon and cluster variables with diagonals. See for example [Sch] for complete details. In [LP, Section 4] we explained how certain cluster algebras could be considered as LP algebras.

Corollary 6.2. *The LP algebra \mathcal{A}'_{P_n} can be identified with the Ptolemy cluster algebra. The identification between seeds and between cluster variables is given by the bijection between maximal nested collections and triangulations.*

6.2. Cycles. Consider the cycle C_n on $[n]$ that consists of edges $i \rightarrow i \pm 1 \pmod n$. The LP algebra \mathcal{A}_{C_n} is the *cycle LP algebra*. Let \mathcal{A}'_{C_n} be the rank $n-1$ LP algebra defined in Section 1.6. The seeds of \mathcal{A}'_{C_n} are exactly the seeds of \mathcal{A}_{C_n} that contain $Y_{[n]}$, and with $Y_{[n]}$ removed.

The strongly connected components P_n are exactly the cyclic intervals $\{[a, b]\}$. There is a bijection between maximal nested families of C_n with support $[n]$ and centrally symmetric triangulations of a $2n$ -gon. The bijection is obtained by mapping a set $[a, b]$ (indices taken modulo n) into the diagonal which cuts vertices a through b away from the rest of vertices. An example is given in Figure 3 where the labeling of the vertices of the $2n$ -gon is shown. The complex of partial centrally symmetric triangulations of the $2n$ -gon give a simplicial complex dual to the cyclohedron.

Theorem 6.3. *Let t_S be a seed of \mathcal{A}'_{C_n} , and let $S_i = IiJ \neq [n]$, where $I, J \in \mathcal{S} \cup \{\emptyset\}$ are chosen so that if $j = i^+$ then $S_j = IiJjK$ for some $K \in \mathcal{S} \cup \{\emptyset\}$ (the concatenated elements and subsets are disjoint and ordered around the cycle). The exchange relation for Y_{S_i} is described as follows:*

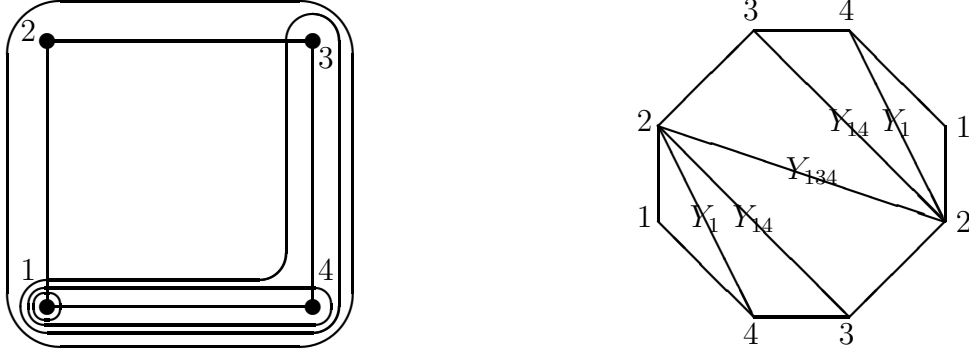


FIGURE 3. An example of the bijection between maximal nested sets on a cycle and centrally symmetric triangulations.

- if $S_{j+} \neq [n]$, we have

$$Y_{IiJ}Y_{JjK} = Y_JY_{IiJjK} + Y_IY_K;$$

- if $S_{j+} = [n]$, so that $IiJj = [n]$, we have

$$Y_{IiJ}Y_{JjI} = Y_IY_JY_{[n]} + (Y_I + Y_J)^2.$$

Proof. Direct application of (5.2). \square

The LP algebra \mathcal{A}'_{C_n} cannot be identified with a type B cluster algebra, even though it has the same cluster complex (dual to the cyclohedron).

Example 6.4. The seed corresponding to the maximal nested collection in Figure 3 has the following form:

$$\{(Y_{134}, (1 + Y_{14})^2 + Y_{1234}Y_{14}), (Y_{14}, Y_{134} + Y_1), (Y_1, 1 + Y_{14})\}.$$

In particular, we see that some of the exchange polynomials are not binomial, and thus this cannot be identified with a type B cluster algebra.

6.3. Complete graphs. Consider the complete graph K_n on $[n]$ that consists of all edges $i \rightarrow j$ for $i \neq j$. The clusters are in bijection with activation sequences $\vec{S} = (s_1, \dots, s_k)$, each sequence yielding a different cluster.

For $1 \leq j \leq k$ define $\mathcal{C}_{S_{s_j}}$ by

$$\mathcal{C}_{S_{s_j}} = \prod_{\ell=1}^{j-1} Y_{S_{s_\ell}}^{2^{j-\ell-1}}$$

and define $\mathcal{P}_{S_{s_j}}$ recursively via

$$\mathcal{P}_{S_{s_1}} = 1 + Y_{s_1} \quad \text{and} \quad \mathcal{P}_{S_{s_j}} = \prod_{1 \leq \ell < j} \mathcal{P}_{S_{s_\ell}} + Y_{S_{s_j}} \mathcal{C}_{S_{s_{j-1}}}.$$

For example, $\mathcal{P}_{S_{s_4}} =$

$$(1 + Y_{s_1})(1 + Y_{s_1} + Y_{s_1s_2})((1 + Y_{s_1})(1 + Y_{s_1} + Y_{s_1s_2}) + Y_{s_1}Y_{s_1s_2s_3}) + Y_{s_1}^2Y_{s_1s_2}Y_{s_1s_2s_3s_4}.$$

Theorem 6.5. The exchange polynomials of the seed $t_{\vec{S}}$ of \mathcal{A}_{K_n} are given by

- For $j < k$ we have

$$F_{S_{s_j}} = \prod_{1 \leq \ell < j} \mathcal{P}_{S_{s_\ell}}^2 + Y_{S_{s_{j+1}}} \mathcal{C}_{S_{s_j}} \quad \text{and} \quad \hat{F}_{S_{s_j}} = \frac{F_{S_{s_j}}}{\mathcal{C}_{S_{s_{j-1}}}^2}.$$

- For $j = k$ we have

$$F_{S_{s_k}} = A_{s_k} Y_{S_{s_{k-1}}} \mathcal{C}_{S_{s_{k-1}}} + \sum_{1 \leq j < k} (A_{s_j} Y_{S_{s_{j-1}}} \mathcal{C}_{S_{s_{j-1}}} \cdot \prod_{1 \leq \ell < k, \ell \neq j} \mathcal{P}_{S_{s_\ell}}) + \left(\sum_{j \notin S} X_j \right) \prod_{1 \leq \ell < k} \mathcal{P}_{S_{s_\ell}}$$

$$\text{and} \quad \hat{F}_{S_{s_k}} = \frac{F_{S_{s_k}}}{\mathcal{C}_{S_{s_{k-1}}}}.$$

- For $m \notin S$ we have

$$F_m = A_m Y_{S_{s_k}} \mathcal{C}_{S_{s_k}} + \sum_{1 \leq j \leq k} (A_{s_j} Y_{S_{s_{j-1}}} \mathcal{C}_{S_{s_{j-1}}} \cdot \prod_{1 \leq \ell \leq k, \ell \neq j} \mathcal{P}_{S_{s_\ell}}) + \left(\sum_{j \notin S, j \neq m} X_j \right) \prod_{1 \leq \ell \leq k} \mathcal{P}_{S_{s_\ell}}$$

$$\text{and} \quad \hat{F}_m = \frac{F_m}{\mathcal{C}_{S_{s_k}}}.$$

The proof is meticulous but straightforward, and it is omitted.

Example 6.6. Take $n = 4$ and $\vec{S} = (2, 4, 1)$. Then the variables are $(Y_{124}, Y_2, X_3, Y_{24})$, and the seed $t_{\vec{S}}$ looks like

$$\{(Y_{124}, A_1 Y_{24} Y_2 + A_4 Y_2 (1 + Y_2) + A_2 (1 + Y_2 + Y_{24}) + X_3 (1 + Y_2) (1 + Y_2 + Y_{24})),$$

$$(Y_2, 1 + Y_{24}),$$

$$(X_3, A_3 Y_{124} Y_{24} Y_2^2 + A_1 Y_{24} Y_2 (1 + Y_2) (1 + Y_2 + Y_{24}) + A_4 Y_2 (1 + Y_2) ((1 + Y_2) (1 + Y_2 + Y_{24}) + Y_{124} Y_2) +$$

$$A_2 (1 + Y_2 + Y_{24}) ((1 + Y_2) (1 + Y_2 + Y_{24}) + Y_{124} Y_2)),$$

$$(Y_{24}, (1 + Y_2)^2 + Y_{124} Y_2)\}.$$

7. CONJECTURES

The following are some questions and conjectures about graph LP algebras and more generally linear LP algebras.

Conjecture 7.1. *Let \mathcal{A}_Γ be a graph LP algebra and t_S a seed. Then*

- (1) *All the exchange polynomials F_i of t_S are positive polynomials in $\mathcal{L}(t_S)$.*
- (2) *Any cluster variable of \mathcal{A}_Γ has positive coefficients when written as a Laurent polynomial in $\mathcal{L}(t_S)$.*

The Laurentness of Conjecture 7.1(2) follows from Theorem 2.2. Conjecture 7.1(2) implies Conjecture 7.1(1). For the case $t_S = t_\emptyset$ is the initial seed, Conjecture 7.1(2) follows from the definition of the Y_I . The calculations in Section 6 also support Conjecture 7.1.

Problem 7.2. *Find a combinatorial interpretation for the coefficients of the Laurent polynomials in the Conjecture 7.1.*

A monomial in the cluster variables of \mathcal{A}_Γ is a *cluster monomial* if there is some seed t_S containing all the variables occurring in that monomial.

Conjecture 7.3. *Let \mathcal{A}_Γ be a graph LP algebra.*

- (1) *Cluster monomials form a basis for \mathcal{A}_Γ over R .*
- (2) *Any monomial in the cluster variables of \mathcal{A}_Γ can be written as a nonnegative linear combination of cluster monomials.*

Problem 7.4. *Find a combinatorial interpretation for the coefficients of cluster monomials in Conjecture 7.3.*

In this paper we have focused on exchange polynomials of the form $F_i = A_i + \sum_{i \rightarrow j} X_j$. Allowing the A_i to be specialized, or the X_j to have coefficients gives an arbitrary linear LP algebra.

Conjecture 7.5. *Any linear LP algebra is of finite type.*

Conjectures 7.1 and 7.3 should also have analogues for linear LP algebras whose coefficients satisfy a positivity constraint.

Our results have established that the cluster complex of a graph LP algebra \mathcal{A}_Γ is isomorphic to the extended nested complex $\tilde{\mathcal{N}}(\Gamma)$. We conjecture that $\tilde{\mathcal{N}}(\Gamma)$ is dual to a the face lattice of a polytope. More generally,

Conjecture 7.6. *Let \mathcal{A} be any linear LP algebra. Then there exists a convex simple polytope $P(\mathcal{A})$ of dimension equal to the rank of \mathcal{A} such that*

- (1) *the vertices of $P(\mathcal{A})$ are in bijection with clusters of \mathcal{A}*
- (2) *the facets of $P(\mathcal{A})$ are in bijection with cluster variables of \mathcal{A}*

and under these bijections we have that for each face F of $P(\mathcal{A})$ the collection of facets containing F are exactly the set of cluster variables contained in the intersection of the clusters corresponding to the vertices of F .

In particular, Conjecture 7.6 would imply that the exchange graph of any linear LP algebra is the 1-skeleton of a polytope. The graph associahedra of Carr and Devadoss [CD] and more generally the nestohedra of Postnikov [Pos] show that the LP algebras \mathcal{A}_Γ satisfy Conjecture 7.6.

8. PROOF OF THEOREM 2.4

For polynomials P, Q and a variable x , define $\text{den}(P, x, Q)$ to be

$$\text{den}(P, x, Q) = \min_i (i + \max(s : Q^s | p_i))$$

where $P(x) = \sum_{i=0}^r p_i x^i$. By definition we have

$$\hat{F}_i = \prod_{j \neq i} x_j^{-\text{den}(F_i, x_j, F_j)} F_i.$$

Note that $\text{den}(P, x, Q)$ makes sense even when P is a Laurent polynomial in some variables, as long as Q is not divisible by any of these variables.

Lemma 8.1. *Suppose Q is irreducible. Then*

$$\text{den}(P, x, Q) + \text{den}(P', x, Q) = \text{den}(PP', x, Q).$$

Proof. The inequality \leq is clear. Let $d = \text{den}(P, x, Q) + \text{den}(P', x, Q)$. Let i_0 be the minimal i such that the minimum $\min_i(i + \max(s : Q^s | p_i))$ is achieved, and similarly for j_0 and $\min_j(j + \max(s : Q^s | p'_j))$. Set $k = i_0 + j_0$. Then the coefficient of x^k in PP' is $p_0 p'_k + \cdots + p_{i_0} p'_{j_0} + \cdots + p'_k p_0$. By the minimality assumption, Q^{d-k+1} divides all the terms except for $p_{i_0} p'_{j_0}$, which is only divisible by Q^{d-k} . Thus $\text{den}(PP', x, Q) \leq d$. \square

Start with a seed $t = \{(a, P), (b, Q), (x_1, F_1), \dots, (x_k, F_k)\}$. We make the assumption that P may depend on b , and that Q does not depend on a . Furthermore, we assume that P/Q is not a unit, so in particular that $\text{den}(Q, a, P) = 0$. We need to construct the successive mutations of t at a and b . In the case that we only have two cluster variables, the claim we want was established in [LP, Proposition 6.3].

8.1. We mutate the seed t at b to get $\{(a, R), (b', Q), \dots, (x_i, F'_i), \dots\}$. Our first calculation is that $\hat{P} = \hat{R}$ after we substitute $b' = \hat{Q}/b$. Strictly speaking, we mean that we can choose the mutation at b so that $\hat{P} = \hat{R}$, since R is only defined up to units. We will abuse notation in this way throughout.

First we may write

$$Q = \hat{Q} \times \prod_i x_i^{\text{den}(Q, x_i, F_i)} = \hat{Q} \times D.$$

Write

$$P(b) = p_0 + p_1 b + \cdots + p_r b^r.$$

Then since Q does not depend on a , we have

$$R(b') = \frac{p_0 (Db')^r + p_1 (Db')^{r-1} Q + \cdots + p_r Q^r}{Q^s E} = P(\hat{Q}/b') \frac{(Db')^r}{Q^s E}$$

for some $0 \leq s \leq r$ and E some monomial in the x_i . Also we have

$$\hat{P} = \frac{P}{b^s \prod_i x_i^{\text{den}(P, x_i, F_i)}}$$

and from the calculation in [LP, Proposition 6.3], we already know that $\text{den}(R, b, Q) = r - s$. For each i define $d_i = \text{den}(Q, x_i, F_i)$. Note that the monomial E above only involve x_i such that $d_i > 0$. Let $R' = \frac{P(\hat{Q}/b')(Db')^r}{Q^s}$.

Lemma 8.2. *We have $\text{den}(R', x_i, F'_i) = d_i(r - s) + \text{den}(P, x_i, F_i)$.*

Proof. Case 1: Suppose F_i does not depend on b . Then $F'_i = F_i$. Let $R'' = P(\hat{Q}/b')(Db')^r$. Then

$$\text{den}(R'', x_i, F_i) = \min_j (d_i(r-j) + \text{den}(Q^j p_j, x_i, F_i)) = \min_j (d_i r + \text{den}(p_j, x_i, F_i)) = d_i r + \text{den}(P, x_i, F_i).$$

Thus $\text{den}(R', x_i, F'_i) = \text{den}(R''/Q^s, x_i, F_i) = d_i(r - s) + \text{den}(P, x_i, F_i)$.

Case 2: Suppose F_i depends on b . Then $d_i = 0$ and F'_i depends on b' . Also x_i does not occur in D or E . We shall prove that $\text{den}(P, x_i, F_i) = \text{den}(R, x_i, F'_i)$ which suffices since $\text{den}(R, x_i, F'_i) = \text{den}(R', x_i, F'_i)$. By symmetry, it is enough to show that $\text{den}(R, x_i, F'_i) \geq \text{den}(P, x_i, F_i)$. Let $P = a_0 + a_1 x_i + \cdots + a_k x_i^k$. If F_i^s divides a_j then we have $a_j = F_i^s G$ and

$$a_j(\hat{Q}/b') = F_i(\hat{Q}/b')^s G(\hat{Q}/b') = (F_i(\hat{Q}(0)/b') + x_i H)^s G(\hat{Q}/b')$$

where $\hat{Q}(0)$ is the evaluation of \hat{Q} at $x_i = 0$, and H is a polynomial in x_i . Since F'_i is a divisor of $F_i(\hat{Q}(0)/b')$ up to a monomial in the other variables, we see that $\text{den}(a_j(\hat{Q}/b'), x_i, F'_i) \geq s$. It follows that $\text{den}(R, x_i, F'_i) \geq \text{den}(P, x_i, F_i)$. \square

Using the Lemma 8.2, we calculate

$$\begin{aligned} \hat{R} &= \frac{R'}{b'^{r-s} \prod_i x_i^{\text{den}(R', x_i, F'_i)}} \\ &= \frac{P(Db')^r}{b'^{r-s}(Db'b)^s \prod_i x_i^{\text{den}(R', x_i, F'_i)}} \\ &= \frac{P \prod_i x_i^{d_i(r-s)}}{b^s \prod_i x_i^{\text{den}(R', x_i, F'_i)}} \\ &= \frac{P}{b^s \prod_i x_i^{\text{den}(P, x_i, F_i)}} = \hat{P} \end{aligned}$$

Given the seed t , we can mutate at a , then b , then at the new variable a' at the position of a , then at the new variable b' at the position at b . We can also mutate first at b , then a , then b' , then a' . In the following, we call this “mutating four times”.

Lemma 8.3. *Mutating four times we recover the cluster variable a .*

Proof. We have shown that $\hat{P} = \hat{R}$. Thus if we mutate $\{(a, R), (b', Q), \dots\}$ at a to get $\{(a', R), (b', S), \dots\}$ the variable $a' = \hat{R}/a = \hat{P}/a$ is the same as what we get if we mutate $\{(a, P), (b, Q), \dots\}$ at a . But mutating one more time at the position of b (on either end) will not change the first cluster variable. This is true for all seeds t' with the the exchange polynomial of b does not depend on a . The claim follows. \square

8.2. Our next task is to show that mutating four times preserves b . We first mutate $\{(a, P), (b, Q), \dots\}$ at a to get a seed $\{(a', P), (b, S), \dots\}$. But Q does not depend on a , so we may pick $S = Q$. We now show that $\hat{Q} = \hat{S}$, which would follow from $\text{den}(Q, x_i, F_i) = \text{den}(Q, x_i, F'_i)$ where now F'_i is the exchange polynomial obtained by mutation at a . Suppose first that F_i depends on a , implying that F'_i also depends on a . Then since Q does not depend on a , we have $\text{den}(Q, x_i, F_i) = 0 = \text{den}(Q, x_i, F'_i)$. Otherwise F_i does not depend on a , and so $F_i = F'_i$, and we have $\text{den}(Q, x_i, F_i) = \text{den}(Q, x_i, F'_i)$. The same argument as in Lemma 8.3 gives

Lemma 8.4. *Mutating four times we recover the cluster variable b .*

8.3. We now show that the exchange polynomials P and Q are recovered after four mutations. For Q that is clear since mutation at a or b does not change Q . Now consider the sequence of 3 mutations at b, a, b' , giving $\{(a, R), (b', Q), \dots\}$, then $\{(a', R), (b', Q), \dots\}$, then $\{(a', T), (b, Q), \dots\}$. We need to show that $T = P$. But we know that one more mutation at a gives us $\{(a, T), (b, Q), \dots\}$, using the fact that a is recovered after four mutations. But we thus have $a' = \hat{T}/a = \hat{P}/a$ where \hat{T} can be calculated in the seed $\{(a', T), (b, Q), \dots\}$, while \hat{P} is calculated in the seed $\{(a, P), (b, Q), \dots\}$. Here we have used that a' is also recovered after four mutations. Thus $\hat{T} = \hat{P}$. Both are Laurent polynomials in b and x_i , so we must have $T = P$ as polynomials in b and x_i .

8.4. Finally we need to show that (x_i, F_i) is recovered after four mutations. This is clear for x_i . Note that it is enough to recover F_i up to a monomial factor in the variables a, b, x_j , so we can just work with Laurent polynomials throughout. Consider the sequence of seeds $t_1 = \{(a, P), (b, Q), \dots\}$, $t_2 = \mu_b(t_1) = \{(a, R), (b', Q), \dots\}$, then $t_3 = \mu_a(t_2) = \{(a', R), (b', Q), \dots\}$, then $t_4 = \mu_{b'}(t_3) = \{(a', P), (b, Q), \dots\}$, then $t_5 = \mu_{a'}(t_4) = \{(a, P), (b, Q), \dots\}$. Fix i , and let the corresponding F_i in each seed be denoted

$$Z_1(a, b), Z_2(a, b'), Z_3(a', b'), Z_4(a', b), Z_5(a, b).$$

We need to show that $Z_1 = Z_5$. We shall first assume that Z_1 depends on a and b . We have (up to a monomial factor in a, b, x_j)

$$Z_2(a, b') = Z_1(a, \hat{Q}(0)/b')/A$$

where A is some product of factors of $\hat{Q}(0)$ not depending on a, b', x_i . Here $\hat{Q}(0)$ is the evaluation of $\hat{Q}(x_i)$ at $x_i = 0$. Similarly,

$$Z_3(a', b') = Z_2(\hat{R}(0)/a', b')/B(b') = \frac{Z_1(\hat{R}(0)/a', \hat{Q}(0)/b')}{AB(b')}$$

where since Z_2 is defined up to a monomial factor, so is Z_3 up to a monomial factor in a', b', x_j , and in addition $B(b')$ is defined up to a power of $\hat{R}(0)$ since Z_2 was defined up to a power of a . Continuing,

$$Z_4(a', b) = \frac{Z_1(\hat{R}(0)/a', b)}{AB(\hat{Q}(0)/b)C}$$

where we have used that \hat{Q} is well-defined regardless of the seed it is calculated in, and we have the usual ambiguities. In addition, the substitution $b = \hat{Q}(0)/b'$ is also made in $\hat{R}(0)$. (Note that it is possible for $Z_3(a, b)$ to not depend on b , but the argument continues in essentially the same way.) Finally,

$$Z_5(a, b) = \frac{Z_1(\frac{\hat{R}(0)}{\hat{P}(0)/a}, b)}{AB(\hat{Q}(0)/b)CD(b)}$$

Now we already showed previously that $\hat{R} = \hat{P}$, so we have that $Z_5(a, b)$ and $Z_1(a, b)$ are equal up to a monomial factor in a, b, x_j , together with the many factors A, B, C, D none of which depend on a , and which are only defined up to certain factors which also do not depend on a . Since $Z_1(a, b)$ depends on both a and b , we must have $Z_5(a, b) = Z_1(a, b)$.

In the case that $Z_1(a, b)$ does not depend on b but does depend on a the argument is similar. When $Z_1(a, b)$ does not depend on a at all, the argument is straightforward.

This completes the proof of Theorem 2.4.

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